Discrete Balayage and Boundary Sandpile

Hayk Aleksanyan

KTH Royal Institute of Technology

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joint work with
Henrik Shahgholian

Lattice growth models, internal DLA - an example

- Place $n \in \mathbb{N}$ identical particles at the origin of $\mathbb{Z}^d$ $(d \geq 2)$.

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- **Question:** What can we say about $V_n \subset \mathbb{Z}^d$ - the (random) set of occupied sites ?

internal DLA

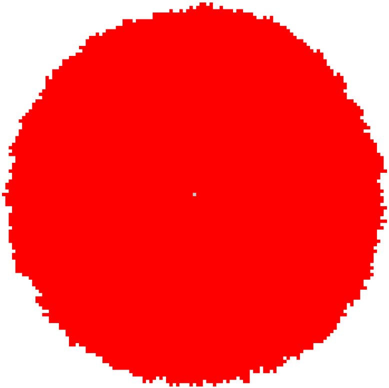

Figure: An occupied cluster on $\mathbb{Z}^2$ for $n = 10\ 000$.

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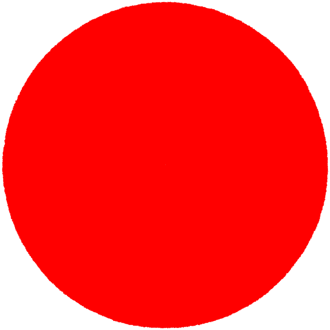

Figure: An occupied cluster on $\mathbb{Z}^2$ for $n = 1\ 000\ 000$.

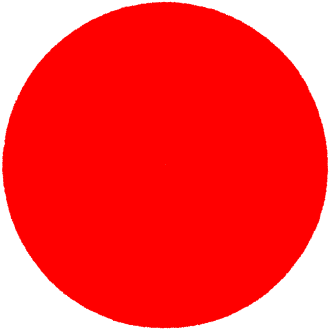

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Figure: Closer look near the north pole.

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internal DLA - scaling limit

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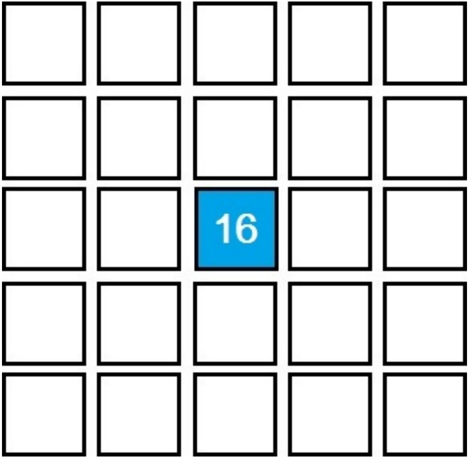
both are **independent** of the order in which the unstable sites are being toppled.

ASM: a toy example



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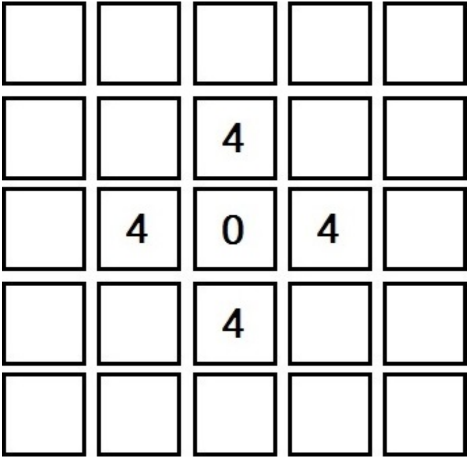

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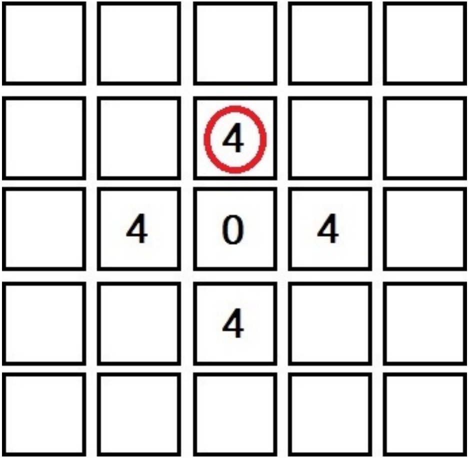

ASM: a toy example

		1		
	1	0	1	
	4	1	4	
		4		

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		1		
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1	0	2	4	
	1	4		

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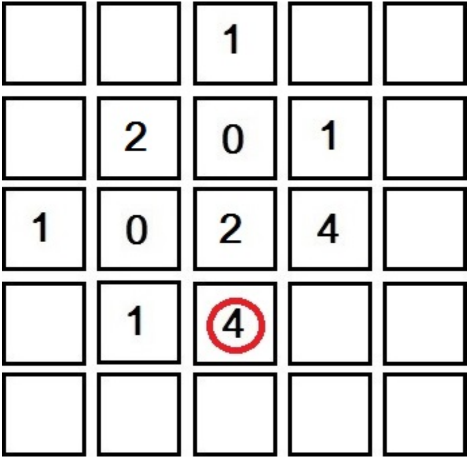

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There is a wealth of mathematics at the heart of this model.

- Viscosity theory of elliptic PDEs
- Appolonian circle packings
- Tropical geometry
- Sandpile groups
- etc.

ASM: large cluster

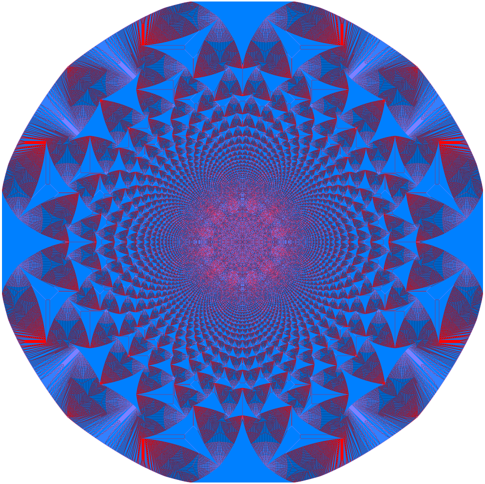

Figure: The limiting shape of the two-dimensional ASM on $\mathbb{Z}^2$ with initial 10 000 000 particles placed at the origin. Sites of $\mathbb{Z}^2$ having 0,1,2, or 3 number of chips are coloured by black, purple, red, and blue respectively.

The odometer and discrete Laplacian

Odometer

For each $x \in \mathbb{Z}^d$ let $u(x) =$ number of times x has toppled during the entire life-time of the sandpile.

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Hence for initial configuration of $n\delta_0$ chips, the final state $s : \mathbb{Z}^d \to \{0, 1, ..., 2d-1\}$ is given by

$$\Delta^1 u(x) = s(x) - n\delta_0.$$

Least action principle

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If s is the stable configuration for n chips, and $u : \mathbb{Z}^d \to \mathbb{Z}_+$ is the odometer then we have

$$n\delta_0 + \Delta^1 u(x) \leq 2d - 1, \qquad x \in \mathbb{Z}^d.$$

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Least Action principle (Fey-Levine-Peres; J. Stat. Phys. '10)

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Each vertex does the **minimal** amount of work for stabilization.

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- For fixed $h > 0$ and each $x \in h\mathbb{Z}^d$ extend u_h as $u_h \equiv u_h(x)$ in the cube $\left[x_1 - \frac{h}{2}, x_1 + \frac{h}{2}\right) \times ... \times \left[x_d - \frac{h}{2}, x_d + \frac{h}{2}\right)$.

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- This extension preserves the discrete Laplacian, we still get $\Delta^h u_h(x) = s_h(x) - h^{-d}\delta_0$ but now for all $x \in \mathbb{R}^d$.

ASM: the scaling limit

Existence of a scaling limit (Pegden-Smart, Duke '13)

There exist compactly supported, non-negative $u_0 \in C(\mathbb{R}^d \setminus \{0\})$, and $s \in L^\infty(\mathbb{R}^d)$ such that $u_h \to u_0$ locally uniformly in $\mathbb{R}^d \setminus \{0\}$ and $s_h \to s$ weak* in $L^\infty(\mathbb{R}^d)$. Moreover, $\Delta u_0 = s - \delta_0$ in a sense of distributions, $0 \leq s \leq 2d - 1$ and $\int_{\mathbb{R}^d} s(x)dx = 1$.

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Relaxing the integral constraint in ASM

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A "slight" change in the obstacle problem for ASM

$$u(x) := \inf\{w : \mathbb{Z}^d \to \mathbb{R}_+ : \ n\delta_0 + \Delta^1 w(x) \leq 1\},$$

where $n > 0$ is a continuous mass now, and

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A full site can *topple* by **evenly** distributing its excess from 1 among its $2d$ lattice neighbours.

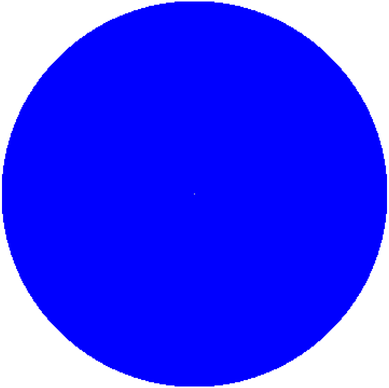

Figure: The redistribution of mass 100 000 by divisible sandpile.

Divisible sandpile and Quadrature domains

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Theorem (Levine-Peres [J. Anal. Math, '10])

For a "sufficiently nice" initial density the scaling limit of the divisible sandpile is a quadrature domain.

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Generate a quadrature surface via sandpile dynamics.

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$$\int h(x) d\mu(x) = \int_{\partial D} h(x) d\mathcal{H}_{\partial D},$$

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Expressed differently (Shahgholian, [Ark. Math. '94]) for a given measure μ find a domain D where the problem

$$\Delta u = -\mu \text{ in } D, \; u = 0 \text{ on } \partial D \text{ and } \frac{\partial u}{\partial \nu} = -1 \text{ on } \partial D,$$

has a solution.

Boundary sandpile: definition of the process $\mathrm{BS}(\mu_0, \kappa_0)$

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Fix a mass distribution $\mu_0 : \mathbb{Z}^d \to \mathbb{R}_+$ with *finite support* and *bounded total mass*, and set a threshold $\kappa_0 > 0$.

Boundary sandpile: definition of the process $\mathrm{BS}(\mu_0, \kappa_0)$

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- Topple an unstable site by evenly distributing **all** its mass equally among its $2d$ lattice neighbours.
- If there are no unstable sites, stop (will **never** stop except for trivial cases).
- V_k shows the set of visited sites, $u_k(x)$ is the amount of emissions from $x \in \mathbb{Z}^d$, and μ_k is the distribution - all computed after the k-th toppling has been invoked (times before k are included).

For all $k = 0, 1, \ldots$ we get $\Delta^1 u_k(x) = \mu_k(x) - \mu_0(x)$, $x \in \mathbb{Z}^d$.

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Existence of stable states

Let $T = \{x_k\}_{k=1}^{\infty} \subset \mathbb{Z}^d$ be any, s.t. all $x \in \mathbb{Z}^d$ appear in T infinitely often. Then, one-by-one toppling vertices of T produces a stable configuration in the limit.

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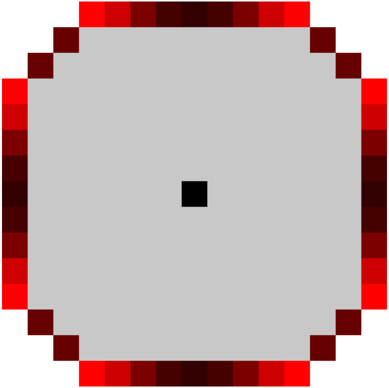
Note: Abelian property is **NOT** automatic; there are non-Abelian sandpiles [Fey-den Boer, Liu; J. Cell. Automata '11].

Boundary sandpile: how does it look like?



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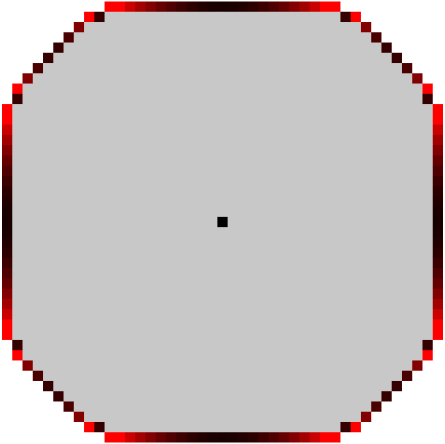

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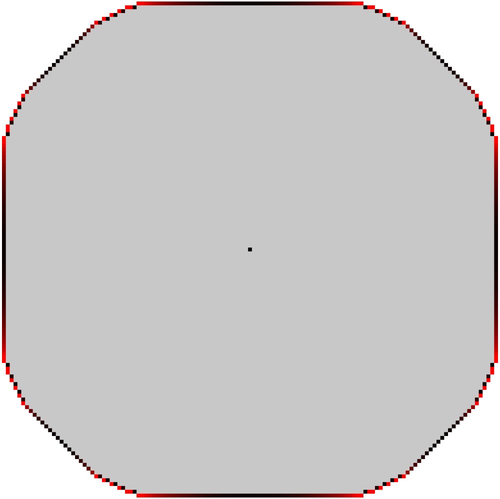

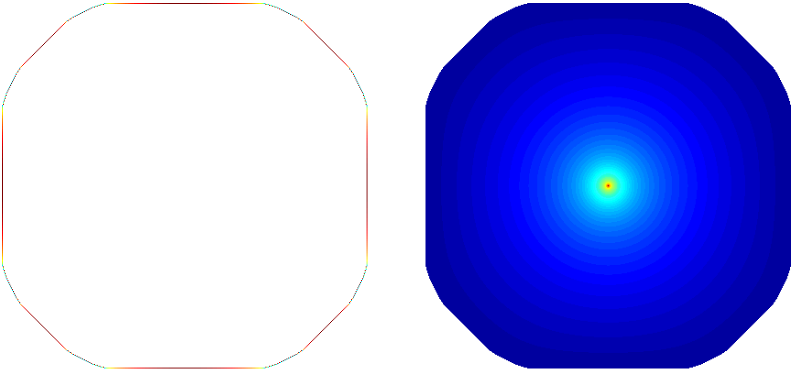

Figure: Final configuration of the BS on the left, with initial mass of 1 000 000 concentrated at the origin of $\mathbb{Z}^2$ and boundary capacity equal to 1 000. The odometer is on the right. Warmer colors (starting from dark red) have larger numerical values than the cooler ones (terminating at dark blue). The odometer develops a singularity at the source.

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Let V_0 be the set of visited sites for $\mathrm{BS}(\mu_0, \kappa_0)$. Then, V_0 is the **intersection** *of all $V \subset \mathbb{Z}^d$ for which there is a function u s.t. the pair (V, u) is stabilizing.*

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Reformulation in terms of a discrete obstacle problem

If u is the odometer for $\mathrm{BS}(\mu_0, \kappa_0)$ then

$$u = \inf\{w : \mathbb{Z}^d \to \mathbb{R}_+ : \quad \mu_0 + \Delta^1 w \leq \kappa_0 \mathbb{I}_{\partial\{w>0\}} \text{ on } \mathbb{Z}^d\}.$$

Growth of the model for a single source

Consider $\mathrm{BS}(n\delta_0, \kappa_0)$ and let (V, u) be **the stabilizing pair**.

Comparison with sub-solutions

Fix $B \subset V$ with origin in its interior and let $v : B \to \mathbb{R}$ satisfy

$$\Delta^1 v \geq -n\delta_0 \text{ in } \overset{\circ}{B}, \;\; v = 0 \text{ on } \partial B \text{ and } \Delta^1 v > \kappa_0 \text{ on } \partial B.$$

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Proof: For $w := u - v$ we get $\Delta^1 w \leq 0$ in $\overset{\circ}{B}$ and $w \geq 0$ on ∂B. *Discrete maximum principle* implies $\min_B w \geq \min_{\partial B} w \geq 0$. Hence, if $\exists\ x_0 \in \partial B \cap \partial V$, then $\kappa_0 < \Delta^1 v(x_0) \leq \Delta^1 u(x_0) \leq \kappa_0$, a contradiction. $\square$

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To estimate the growth of the visiting sites, compare it with a controllable sub-solution.

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For $R > 0$ let G_R be the Green's function (pole at the origin) for the discrete ball Z_R.

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Laplacian of G_R

There exists R_0 depending on dimension d and $0 < c_1 < C_1$ depending on d and R_0 s.t.

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Proof: Write $G_R(x) = g(x,0) - \mathbb{E}^x[g(S_{\tau_R}, 0)]$, where $g(x,y)$ is the fundamental solution of Δ^1 on $\mathbb{Z}^d \times \mathbb{Z}^d$, and τ_R is the first exit time from $\overset{\circ}{Z}_R$ of a simple random walk started at x. Use that the walk will exit Z_R in finite time with probability 1, then apply the asymptotics of g.

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- The *minimality principle* principle implies *monotone and unbounded growth* for a model $\mathrm{BS}(n\delta_0, n^{1/d})$.
- Discrete Harnack implies **non-degeneracy**, i.e. for any $r_0 > 0$ there is a constant $c_0 > 0$ depending on r_0 and d such that for any $n > 1$ and each $x_0 \in \partial V_n$ with $\mathrm{dist}(x_0, \partial V_n) \geq r_0 n^{1/d}$ one has $u_n(x_0) \geq c_0 n^{2/d}$.

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  The meaning of this, is that the odometers do not **flatten out**.

Lipschitz bounds

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Using random walk counterpart of the problem, and known asymptotic bounds for (discrete) fundamental solutions we get

Gradient bounds in annulus

Fix $r_0 > 0$. There is a constant $C = C(r_0) > 0$ such that for any $n > 1$ one has

$$|u_n(x) - u_n(y)| \leq Cn^{1/d},$$

for any $x, y \in V_n$ with $x \sim y$ and $r_0 n^{1/d} \leq |x| \leq 2r_0 n^{1/d}$.

Lipschitz bounds

Uniform Lipschitz estimate away from the origin

For any $r_0 > 0$ there exists a constant $C = C(r_0) > 0$ s.t. for any $n > 1$ and each $x, y \in V_n$ with $|x|, |y| > r_0 n^{1/d}$ one has $|u_n(x) - u_n(y)| \leq C n^{1/d} |x - y|$.

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Proof: The discrete derivative $\partial_i^+ u(x) := u(x + e_i) - u(x)$ is Δ^1-harmonic in

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By discrete maximum principle + gradient bounds in the annulus + stability of the sandpile, we get

$$|\partial_i^+ u(x)| \leq C n^{1/d} \text{ for } x \in V_{n,0}.$$

Now, for any $x, y \in V_n$ fix a path $x = X_0 \sim \dots \sim X_k = y$ through $V_{n,0}$, with $k \asymp |x - y|$, and apply 1-step Lipschitz bound.

Scaled odometers are uniformly Lipschitz away from the origin

Set $h = n^{-1/d}$ and $u_h(x) := h^2 u_n(h^{-1}x)$, $x \in h\mathbb{Z}^d$. Then

$$|u_h(x) - u_h(y)| \leq C_{r_0}|x - y|$$

for all $x, y \in h\mathbb{Z}^d$ with $|x|, |y| > r_0$.

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- This eventually implies convergence of the sandpiles along subsequences as (mesh-size) $h \to 0$.

Convergence

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- This eventually implies convergence of the sandpiles along subsequences as (mesh-size) $h \to 0$.
- We do NOT know if the limit is unique.

Directional monotonicity via moving planes

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Let $\mathcal{S}$ be the set of **mirror symmetry** hyperplanes of the cube $[0,1]^d$, Elements of $\mathcal{S}$ are the hyperplanes (d^2 in total)

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Theorem

Let (V, u) be the stabilizing pair of $\mathrm{BS}(n\delta_0, \kappa_0)$ and $T \in \mathcal{S}$ be any. Then, for any $X_1, X_2 \in \mathbb{Z}^d$ s.t. $X_1 - X_2 \neq 0$ is orthogonal to T we have

$$u(X_1) \geq u(X_2) \iff |X_1| \leq |X_2|.$$

The same is true for Abelian Sandpile model (with a simpler proof).

Directional monotonicity via moving planes

Let $\mathcal{S}$ be the set of **mirror symmetry** hyperplanes of the cube $[0,1]^d$, Elements of $\mathcal{S}$ are the hyperplanes (d^2 in total)

$$\{x_i = 1/2\},\ \{x_i = x_j\} \text{ and } \{x_i = -x_j\} \text{ with } 1 \leq i \neq j \leq d.$$

Theorem

Let (V, u) be the stabilizing pair of $\mathrm{BS}(n\delta_0, \kappa_0)$ and $T \in \mathcal{S}$ be any. Then, for any $X_1, X_2 \in \mathbb{Z}^d$ s.t. $X_1 - X_2 \neq 0$ is orthogonal to T we have

$$u(X_1) \geq u(X_2) \iff |X_1| \leq |X_2|.$$

The same is true for Abelian Sandpile model (with a simpler proof).

The intuition: There is less "action" away from the source.

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$$u_T(x) = \begin{cases} u(x), & \text{if } x \in V_- := \mathcal{H}_- \cap V, \\ \min\{u(x), u^*(x)\}, & \text{if } x \in V_+ := \mathcal{H}_+ \cap V \cap V^*. \end{cases}$$

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- Since $X_2 \in \mathcal{H}_+$ we get $u(X_2) \leq u_T(X_2) = \min\{u(X_1), u(X_2)\} \leq u(X_1)$ q.e.d.

Lipschitz boundary via monotonicity of odometers

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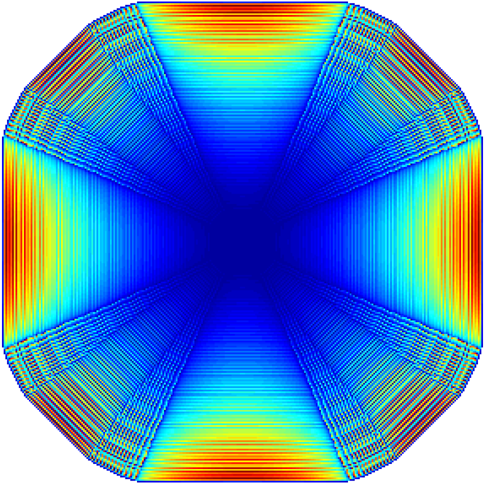
- The (truncated) double-cone spanned by this direction vectors and having vertex at x_0 has no points of the free boundary in its interior.
- Since the free boundary satisfies the double-cone condition, it is a Lipschitz graph locally.

Works the same for Abelian Sandpile, and Boundary Sandpile.

Averaged mass distribution of BS over n=1,..., 500 000



Thank you!