Boundary value homogenization of Dirichlet problem for divergence type elliptic operators

Hayk Aleksanyan

KTH Royal Institute of Technology,

Stockholm, Sweden

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Fixed operator and oscillating Dirichlet data

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$$(\mathcal{L}u)_i = -D_\alpha[A_{ij}^{\alpha\beta}(\cdot)D_\beta u_j] := -\nabla \cdot [A(x)\nabla u(x)].$$

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- $D \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded domain,
- $g(x, y) : \partial D \times \mathbb{R}^d \to \mathbb{C}^N$ is $\mathbb{Z}^d$-periodic in y, i.e.

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The problem:

$$\begin{cases} \mathcal{L}u_\varepsilon = 0 & \text{in } D, \\ u_\varepsilon(x) = g(x, x/\varepsilon) & \text{on } \partial D \;. \end{cases}$$

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Let u_ε be the solution to the problem with fixed operator, and boundary data $g(\cdot, \cdot/\varepsilon)$, and u_0 be the solution to the same problem but with boundary data $\overline{g}(x) = \int_{\mathbb{T}^d} g(x,y)dy$, $x \in \partial D$.

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Theorem (Pointwise estimates; J. Diff. Eq. '13, joint with H. Shahgholian, and P. Sjölin)

For each $\kappa > d-1$ there exists a constant C_κ such that

$$|u_\varepsilon(x) - u_0(x)| \leq C_\kappa \min\left\{1, \frac{\varepsilon^{(d-1)/2}}{d(x)^\kappa}\right\}, \qquad \forall x \in D,$$

where $d(x)$ is the distance of x from the boundary of D.

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Corollary1

For each $1 \leq p < \infty$ and each $\kappa < \frac{1}{2p}$ there exists a constant C_κ such that

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_\kappa \varepsilon^\kappa.$$

...continuing (beyond strict convexity)

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Corollary2

Assume D is a bounded and smooth domain in $\mathbb{R}^d$, such that there is an integer $1 \leq m \leq d-1$ for which at any $x \in \partial D$ *at least* m of the principal curvatures of ∂D are non-zero.

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Then, for each $\kappa > m$ we have

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$$|u_\varepsilon(x) - u_0(x)| \leq C_\kappa \min\left\{1, \frac{\varepsilon^{m/2}}{d(x)^\kappa}\right\}, \qquad \forall x \in D.$$

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$$|u_\varepsilon(x) - u_0(x)| \leq C_\kappa \min\left\{1, \frac{\varepsilon^{m/2}}{d(x)^\kappa}\right\}, \qquad \forall x \in D.$$

(b) For each $1 \leq p < \infty$ and each $\kappa < \frac{1}{2p}$ there exists a constant C_κ such that

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_\kappa \varepsilon^\kappa.$$

...continuing (why strictly convex?)

Claim

Assume $D \subset \mathbb{R}^d (d \geq 2)$ is a bounded domain with smooth boundary, such that the Gaussian curvature of ∂D is *nowhere vanishing*.

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Use Tietze-Nakajima's theorem (1928) to pass from *local* to *global* convexity.

the domain D is strictly convex.

...continuing (non optimality of L^2 bound)

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An observation

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But neither is $\frac{1}{4}$.

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$$||u_\varepsilon - u_0||_{L^p(D)} \leq C_p \begin{cases} \varepsilon^{1/2p}, & d = 2, \\ (\varepsilon |\ln \varepsilon|)^{1/p}, & d = 3, \\ \varepsilon^{1/p}, & d \geq 4. \end{cases}$$

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Theorem (Optimality of L^p-convergence rate; ibid)

Let $N = 1$, and assume that g depends only on its periodic variable. Then for each $1 \leq p < \infty$ there exists a constant C_p independent of ε, such that

$$\|u_\varepsilon - u_0\|_{L^p(D)} \geq C_p \varepsilon^{1/p} \|g - \overline{g}\|_{L^\infty(\mathbb{T}^d)}.$$

From fixed operator to oscillating

Define $P_k^\gamma = x_\gamma(0, ..., 1, ..., 0) \in \mathbb{R}^N$ with 1 in the k-th position, $1 \leq k \leq N$, $1 \leq \gamma \leq d$. Let $\mathcal{L}_\varepsilon^*$ be the adjoint of $\mathcal{L}_\varepsilon$.

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Theorem (homogenization of the oscillating problem; ibid)

Let $d \geq 3$, and assume that $\mathcal{L}_\varepsilon^*(P_k^\gamma) = 0$ for all $1 \leq k \leq N$, and $1 \leq \gamma \leq d$. Then there exists a boundary term g^* so that if u_0 is the solution of the oscillating problem with boundary data g^* then for any $1 \leq p < \infty$ one has

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Set $v_{k,i}^\gamma(x) = (A_{ki}^{\gamma 1}, ..., A_{ki}^{\gamma d})(x)$, $x \in \mathbb{R}^d$, where $1 \leq k, i \leq N$, $1 \leq \gamma \leq d$. Then $\mathcal{L}_\varepsilon^*(P_k^\gamma) \equiv 0$ is equivalent to

$$\mathrm{div}(v_{k,i}^\gamma)(x) = 0, \qquad x \in \mathbb{R}^d,\ 1 \leq k, i \leq N,\ 1 \leq \gamma \leq d.$$

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For scalar equations ($N = 1$) the condition means that rows of A must be *divergence free* vector fields.

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We can compute the homogenized boundary data in this case. Set

$$h(y) := (h_{ij}(y))_{N\times N} = (A^{0,\alpha\beta} n_\alpha(y) n_\beta(y)), \qquad y \in \partial D.$$

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Then for $g^*(y) = (g_i^*(y))_{i=1}^N$ we have

$$g_i^*(y) = h_{ik}(y) n_\alpha(y) n_\beta(y) \sum_{m\in\mathbb{Z}^d} c_m(A_{kj}^{\alpha\beta}) c_{-m}(g_j; y), \qquad y \in \partial D,$$

where $n(y) = (n_\alpha(y))_{\alpha=1}^d$ is the unit outward normal at y.

The proof of lower bounds

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The Poisson kernel P for the operator $-\nabla \cdot A\nabla$ satisfies

$$|P(x,y)| \lesssim \frac{dist(x, \partial D)}{|x-y|^d}, \qquad x \in D,\ y \in \partial D.$$

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Concentration inequality

There are positive constants c_0, C_0 depending on A, D and d only, s.t. for any $\delta > 0$ small and any $\xi \in \partial D$ one has

$$|u(x) - g(\xi)| \leq \frac{1}{8}||g||_{L^\infty} + C_0 \delta Lip(g),$$

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The proof is via integral representation of u and the distance estimate for P.

The proof of lower bounds

Choosing $\delta = a_0\varepsilon$ with $a_0 > 0$ a small constant, we see that

$$|u_\varepsilon(x) - g_\varepsilon(\xi)| \leq \frac{1}{8}||g||_{L^\infty(\mathbb{T}^d)} + C_0 a_0 \varepsilon Lip(g)\frac{1}{\varepsilon} \leq \frac{1}{4}||g||_{L^\infty(\mathbb{T}^d)},$$

for $\forall \xi \in \partial D$ and $\forall x \in D$ satisfying $|x - \xi| \leq a_0\varepsilon$.

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for $\forall \xi \in \partial D$ and $\forall x \in D$ satisfying $|x - \xi| \leq a_0\varepsilon$.

The conclusion

If $|g_\varepsilon(\xi)|$ is large then $|u_\varepsilon(x)|$ remains large in ε-neighbourhood of $\xi \in \partial D$.

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The conclusion

If $|g_\varepsilon(\xi)|$ is large then $|u_\varepsilon(x)|$ remains large in ε-neighbourhood of $\xi \in \partial D$.

We need to understand the distribution of g_ε on ∂D, or equivalently $\frac{1}{\varepsilon}\partial D \mod \mathbb{Z}^d$.

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Equidistribution of scaled surfaces

Let $D \subset \mathbb{R}^d$ be a bounded domain which is strictly convex and has smooth boundary. Then for any ball $B \subset \mathbb{T}^d$ one has

$$|B| = \lim_{\lambda \to \infty} \frac{\sigma\{x \in \partial D : \ \lambda x \mod \mathbb{Z}^d \in B\}}{\sigma(\partial D)}.$$

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For each non-zero $m \in \mathbb{Z}^d$ one has

$$\left| \int_{\partial D} e^{2\pi i \lambda x \cdot m} d\sigma(x) \right| = |\widehat{\sigma}(\lambda m)| \lesssim (\lambda ||m||)^{-(d-1)/2},$$

where the last estimate is due to convexity.

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where the last estimate is due to convexity.

From here (through Fourier expansion) for any $f \in C^\infty(\mathbb{T}^d)$ we get

$$\int_{\mathbb{T}^d} f(x) dx = \frac{1}{\sigma(\partial D)} \int_{\partial D} f(\lambda x) d\sigma(x).$$

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(b) $\int_{\mathbb{T}^d}(F_n(x) - f_n(x))dx \to 0, \qquad n \to \infty.$

Hence, using the case of smooth function proved above, we get

$$\int_{\mathbb{T}^d} \mathbb{I}_B(x)dx = \frac{1}{\sigma(\partial D)} \int_{\partial D} \mathbb{I}_B(\lambda x) d\sigma(x),$$

and we are done.

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Then, for $\varepsilon > 0$ small we have

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$$|u_\varepsilon(x)| \geq |g_\varepsilon(y)| - |u_\varepsilon(x) - g_\varepsilon(y)| \geq \frac{1}{2}||g||_{L^\infty} - \frac{1}{4}||g||_{L^\infty},$$

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$$|u_\varepsilon(x)| \geq |g_\varepsilon(y)| - |u_\varepsilon(x) - g_\varepsilon(y)| \geq \frac{1}{2}||g||_{L^\infty} - \frac{1}{4}||g||_{L^\infty},$$

for all $|x - y| \leq a_0\varepsilon$. Thus, on a fixed portion of an ε-neighbourhood of ∂D we get $|u_\varepsilon| \gtrsim \varepsilon$.

Assume $\int_{\mathbb{T}^d} g(x)dx = 0$. Hence $u_0 = 0$ (the homogenized solution).
Consider the set

$$B = \{x \in \mathbb{T}^d : \ |g(x)| > ||g||_{L^\infty(\mathbb{T}^d)}/2\}.$$

WLOG, we may assume that B is a ball.
Then, for $\varepsilon > 0$ small we have

$$\frac{\sigma\{x \in \partial D : \ (1/\varepsilon)x \mod \mathbb{Z}^d \in B\}}{\sigma(\partial D)} > \frac{1}{2}|B|.$$

Fix $y \in \partial D$ such that $|g_\varepsilon(y)| > ||g||_{L^\infty}/2$. Hence

$$|u_\varepsilon(x)| \geq |g_\varepsilon(y)| - |u_\varepsilon(x) - g_\varepsilon(y)| \geq \frac{1}{2}||g||_{L^\infty} - \frac{1}{4}||g||_{L^\infty},$$

for all $|x - y| \leq a_0\varepsilon$. Thus, on a fixed portion of an ε-neighbourhood of ∂D we get $|u_\varepsilon| \gtrsim \varepsilon$. Integrating on that portion gives the desired lower bound.

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The scheme of the proof: Reduction to local graphs

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$$\mathcal{R}(\partial D - z) \cap B(0, r_0) = \{(y', \psi(y')) : \ |y'| \leq 10 r_0\} \cap B(0, r_0),$$

where $y' = (y_1, ..., y_{d-1}) \in \mathbb{R}^{d-1}$, $\psi(0) = |\nabla \psi(0)| = 0$ and

$$Hess\psi(0) = diag(a_1, ..., a_{d-1}),$$

with $0 < c_0 \leq a_1 \leq ... \leq a_{d-1} \leq C_0$ (lower bound due to convexity).

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(b) $|Hess\psi(y') - Hess\psi(0)| \leq \frac{a_1}{1000d}$ for all $|y'| \leq 100\delta$,

(c) $\nabla \psi : B(0, \delta) \mapsto \mathcal{M}$ is one-to-one and onto for some $\mathcal{M} \supset B(0, K_1 \delta)$.

For $L = \frac{K_1}{4K_2}$ and a function $\varphi \in C_0^\infty(B(z, L\delta))$, where $z \in \partial D$, consider

$$I(x) = \int_{\partial D} P(x,y)[g_\varepsilon(y) - \overline{g}(y)]\varphi(y)d\sigma(y).$$

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Translating the origin onto z and rotating the coordinate system by $\mathcal{R}$, we may assume WLOG, that $z = 0$ and $\mathcal{R} = Id$. Thus, passing to volume integral in I we get

$$I(x) = \int\limits_{|y'|<L\delta} P(x,(y',\psi(y')))[g_\varepsilon - \overline{g}](y',\psi(y'))\widetilde{\varphi}(y',\psi(y'))dy'.$$

g is smooth, hence

$$[g_\varepsilon - \overline{g}](y', \psi(y')) = \sum_{m\in\mathbb{Z}^d\setminus\{0\}} c_m(y', \psi(y'))\exp\left[\frac{1}{\varepsilon} m \cdot (y', \psi(y'))\right].$$

The scheme of the proof: Reduction to oscillatory integrals

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Plugging this expansion into $I(x)$, things are reduced to decay estimates for integrals of the form

$$J(x) = \int_{|y'| < L\delta} P(x, (y', \psi(y'))) \Phi(y', \psi(y')) \exp[\lambda F(y')] dy',$$

as $1/\varepsilon =: \lambda \to \infty$,

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as $1/\varepsilon =: \lambda \to \infty$, where $F(y') = n' \cdot y' + n_d \psi(y')$, $|(n', n_d)| = 1$, and $\Phi = 0$ on $|y'| = L\delta$.

Case 1: $|n'| \geq K_1\delta/2$.

The scheme of the proof: Critical points of the phase

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For $D_\varepsilon = \{x \in D : \mathrm{dist}(x, \partial D) \geq \varepsilon\}$ we obtain

$$\int_{D_\varepsilon} |J(x)| dx \lesssim \lambda^{-2} \int_{|w|\geq\varepsilon} \frac{dw}{|w|^{d+1}} \lesssim \lambda^{-2} \frac{1}{\varepsilon} \lesssim \lambda^{-1}.$$

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hence there is a unique $y_0 \in B(0,\delta)$ s.t. $\nabla\psi(y_0) = -\frac{n'}{n_d}$, so $\nabla F(y_0) = 0$.
Since $Hess\psi(y') \approx Hess\psi(0)$ for $|y'| \leq 100\delta$, by Mean-Value Theorem for all $1 \leq j \leq d-1$ we get

$$|(\partial_j F)(y_0 + z')| \geq c|z_j|,$$

if $|y_0 + z'| \leq 100\delta$ and

$$z_j \in \mathcal{C}_j =: \{z' \in \mathbb{R}^{d-1} : \ |z_j'| \geq \frac{1}{2\sqrt{d-1}}|z'|\}.$$

The scheme of the proof: Critical points of the phase

The cones $\{C_j\}$ cover $\mathbb{R}^{d-1}$, hence there is $\{\omega_j\}_{j=1}^{d-1}$ a partition of unity of $\mathbb{R}^{d-1} \setminus \{0\}$ subordinate to $\mathcal{C}_j$ and consisting of degree 0 homogeneous functions smooth away from 0.

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We now isolate the critical point of F. Fix $h \in C_0^\infty(\mathbb{R}^{d-1})$ s.t. $h(y') = 0$ if $|y'| \geq 2$ and $h(y') = 1$ if $|y'| \leq 1$.

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Split $J(x) := J_1(x) + J_2(x)$, where

$$J_1(x) = \int_{|y_0+z'|<L\delta} h(\lambda^{1/2} z') P(x, z^*) \Phi(y_0 + z') \exp[\lambda F(y_0 + z')] dz',$$

$$J_2(x) = \sum_{j=1}^{d-1} \int_{|y_0+z'|<L\delta} [1 - h(\lambda^{1/2} z')] \omega_j(z') \cdots dz',$$

and $z^* = (y_0 + z', \psi(y_0 + z'))$.

The scheme of the proof: Critical points of the phase

The (possible) critical point of the phase is in J_1.
Using smallness of the support of $h(\lambda^{1/2}\cdot)$ we get

$$\int_{D_\varepsilon} |J_1(x)| dx \lesssim \int\limits_{D_\varepsilon} \int\limits_{|z'| \leq 2\lambda^{-1/2}} \frac{dz'}{|x - z^*|^{d-1}} dx \lesssim \lambda^{-(d-1)/2}.$$

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In J_2 we are away from singularity. Integrating by parts (in the j-th coordinate) twice implies

$$\int\limits_{D_\varepsilon} |J_2^{(j)}(x)| \lesssim \lambda^{-2} \begin{cases} \lambda^{3/2}, & d=2, \\ (\lambda|\ln\lambda|), & d=3, \\ \lambda, & d\geq 4. \end{cases}$$

The proof is completed by observing that $vol(D \setminus D_\varepsilon) \sim \varepsilon$.

Flat geometries

Polygon

We say that D is a polygonal domain in $\mathbb{R}^d$ ($d \geq 2$), if it is bounded by some finite number of hyperplanes, i.e.

$$D = \bigcap_{j=1}^{N} \{x \in \mathbb{R}^d : \ \nu_j \cdot x > c_j\},$$

where $c_j \in \mathbb{R}$ and $\nu_j \in \mathbb{S}^{d-1}$.

Diophantine vector

A vector $\nu = (\nu_1, ..., \nu_d) \in \mathbb{R}^d$ is called Diophantine if there exists $0 < \tau(\nu) < \infty$ and $C > 0$ such that

$$|m \cdot \nu| > \frac{C}{||m||^{\tau(\nu)}},$$

for all $m = (m_1, ..., m_d) \in \mathbb{Z}^d \setminus \{0\}$.
We denote the set of such vectors by $\Omega(\tau, C)$.

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For any $\tau > d - 1$ the set $\bigcup_{C>0} \Omega(\tau, C)$ has full measure in any ball of $\mathbb{R}^d$.

Assumptions

We will only consider the case of scalar equations, i.e. $N = 1$, the matrix of coefficients is $A = (A^{\alpha\beta})$, $1 \leq \alpha, \beta \leq d$, and the operator $\mathcal{L}$ is

$$\mathcal{L}(u) = -D_\alpha(A^{\alpha\beta} D_\beta u).$$

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- (Periodicity) The boundary function g is 1-periodic:

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- (Ellipticity) There exists a constant $c > 0$ such that

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- (Smoothness) The boundary value g and all elements of A are sufficiently smooth.

Choose $\alpha_* > 0$ so that $\pi/(1+\alpha_*)$ be the maximal angle between any two adjacent faces of D.

Pointwise convergence

Choose $\alpha_* > 0$ so that $\pi/(1+\alpha_*)$ be the maximal angle between any two adjacent faces of D.

Theorem (Pointwise estimates; J. Fourier Anal. Appl., '14, joint with H. Shahgholian, and P. Sjölin)

If $\alpha_* > 1$ set $\beta = 1$, otherwise let $0 < \beta < \alpha_*$ be any number. Then for any $\delta > 0$ small we have

$$|u_\varepsilon(x) - u_0(x)| \leq C_\delta \left(\frac{\varepsilon^\beta}{d(x)^{\beta+\delta}} \right)^{\frac{d-1}{d-1+\beta}}, \qquad \forall x \in D.$$

L^p bounds for polygonal domains

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Theorem (L^p-estimates; ibid)

For each $1 \leq p < \infty$, and $\delta > 0$ there exists a constant C depending on p, D, $\mathcal{L}$, δ but independent of $\varepsilon > 0$ such that

$$||u_\varepsilon - u_0||_{L^p(D)} \leq C\varepsilon^{\min\{\gamma, \frac{1}{p}\}-\delta}.$$

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If g depends only on its periodic variable, then we also have

$$||u_\varepsilon - u_0||_{L^p(D)} \geq \varepsilon^{1/p}||g - \overline{g}||_{L^\infty(\mathbb{T}^d)}$$

L^p bounds for polygonal domains

Recall that we choose $\alpha_* > 0$ so that $\pi/(1+\alpha_*)$ is the maximal angle between any two adjacent faces of D. Set

$$\gamma = \frac{(d-1)\min\{1,\alpha_*\}}{d-1+\min\{1,\alpha_*\}}.$$

Theorem (L^p-estimates; ibid)

For each $1 \leq p < \infty$, and $\delta > 0$ there exists a constant C depending on p, D, $\mathcal{L}$, δ but independent of $\varepsilon > 0$ such that

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For large p, the exponent $1/p$ is optimal.

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(A1) for each $1 \leq \alpha, \beta \leq d$ we have $A^{\alpha\beta} \in C^\infty(X)$,

(A2) there exist constants $0 < \lambda \leq \Lambda < \infty$ such that

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For $g \in C^\infty(\mathbb{T}^d)$, and a bounded subdomain $D \subset X$ with C^∞ boundary consider the problem

$$-\nabla \cdot A(x)\nabla u_\varepsilon(x) = 0 \text{ in } D \qquad \text{and} \qquad u_\varepsilon(x) = g(x/\varepsilon) \text{ on } \partial D,$$

where $\varepsilon > 0$ is a small parameter. Let also u_0 be the solution to the homogenized problem.

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(a) $|u_{\varepsilon_k}(x) - u_0(x)| \geq \omega(1/\varepsilon_k), \qquad \forall x \in D', \; k = 1, 2, ...$

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Notice that D is NOT strictly convex.

Thank you!