K-surfaces with free boundaries

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May 31, 2017

joint work with

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Precisely, given a disjoint collection $\Gamma = \{\Gamma_1, ..., \Gamma_m\}$ of codimension 2 submanifolds of $\mathbb{R}^{d+1}$, decide if there exists a K-surface of $\mathbb{R}^{d+1}$ (in general immersed) having Γ as its boundary.

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S.-T. Yau, Problem N26, of his list of open problems '90

What conditions should be imposed on a Jordan curve in $\mathbb{R}^3$ so that it can be a boundary of a disk with a given metric of positive curvature?

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- (M. Ghomi; JDG '17) the torsion of any closed curve in $\mathbb{R}^3$ bounding a simply connected locally convex surface vanishes at least 4 times
  - answers a question of H. Rosenberg from 1993
  - is a far reaching extension of the classical 4-vertex theorem, in particular extends the 4-vertex theorem of V.D. Sedykh from 1994

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  If $\Gamma \subset \mathbb{R}^3$ is a smooth curve that projects one-to-one onto $\partial\Omega$, for some Ω smooth, strictly convex planar domain, then Γ bounds a K-surface that is a graph over Ω provided $K > 0$ is small enough.

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- (B. Guan, J. Spruck; Annals 1993) For each integer $n > 0$, there exists an embedded K-surface of **genus** n.
- Further remarkable results by [B. Guan, J. Spruck; JDG 2002, 2004, M. Ghomi; JDG 2001, and other authors...]

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Fix $\Gamma = \{\Gamma_1, ..., \Gamma_m\}$ a collection of disjoint $(d-1)$-dimensional closed smooth embedded submanifolds of $\mathbb{R}^{d+1}$, and let T_0 be a smooth embedded submanifold in $\mathbb{R}^{d+1}$ of codimension 1. Fix also an angle $\theta > 0$.

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What conditions should be imposed on Γ, T_0, and θ in order to get a K-surface spanning Γ and hitting T_0 at an angle θ ?

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- (The target manifold) $T_0 = \mathbb{R}^d \times \{0\}$.
- (The hitting angle) $\theta = \arccos(1 + \lambda_0)^{-1/2}$, for some $\lambda_0 > 0$.

Our setting: formal statement

For a convex domain $\Omega \subset \mathbb{R}^d \times \{0\}$ and parameters $h_0, \lambda_0 > 0$, $K_0 \geq 0$, find a concave function $u : \mathbb{R}^d \times \{0\} \to \mathbb{R}_+$ such that

$$\begin{cases} \det D^2(-u) = K_0 \psi(|\nabla u|), & \text{in } \{u > 0\} \setminus \overline{\Omega}, \\ u = h_0, & \text{on } \partial\Omega, \\ |\nabla u| = \lambda_0, & \text{on } \Gamma_u \end{cases}$$

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For p-Laplace equation, see [A. Henrot and H. Shahgholian; J. Reine und Angew. Math 2000], although methods and motivation are entirely different here.

Weak solutions (à la A.D. Aleksandrov)

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We call a convex $u : E \to \mathbb{R}$ a solution to $\det D^2 u = K_0 \psi(|\nabla u|)$ on E, if for any Borel set $B \subset E$ one has $\int\limits_{\omega_B(u)} \frac{d\xi}{\psi(|\xi|)} = K_0 |B|$.

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The MA measure is weakly* continuous.

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$$u(x_0+t_1\nu) = u\left(\left(1-\frac{t_1}{t_2}\right)x_0 + \frac{t_1}{t_2}(x_0+t_2\nu)\right) \geq \frac{t_1}{t_2}u(x_0+t_2\nu).$$

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- Hence, there is one free parameter, **the slope** of H.
- The **extreme** H (i.e. the "most inclined on the graph") must have slope λ_0.

The main results: homogeneous case

Theorem A ($K_0 = 0$, the homogeneous case)

Let $K_0 = 0$, and $\Omega \subset \mathbb{R}^d$ be bounded convex $C^{1,1}$-regular domain. Then, there exists a unique weak solution u.

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- u is $C^{1,1}$ on $\{u > 0\} \setminus \overline{\Omega}$,
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- if in addition, Ω is strictly convex, then so is the free boundary.

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Example (truncated cone)

Take $\Omega = B(x_0, r)$ in $\mathbb{R}^d$ ($d \geq 2$). Fix $\lambda_0 > 0$ and $h_0 = 1$. Then

$$u(x) = 1 + \lambda_0 - \frac{\lambda_0}{r}|x - x_0|, \quad r \leq |x - x_0| \leq r\left(1 + \frac{1}{\lambda_0}\right)$$

is the solution, with free boundary $|x - x_0| = r(1 + 1/\lambda_0)$.

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Then, there exists a **small** constant $K = K(\Omega, \psi, \lambda_0) > 0$, such that for any $K_0 \in (0, K)$ there exists a weak solution u, which is C^∞ on $\{u > 0\} \setminus \overline{\Omega}$ and the free boundary Γ_u is C^∞ as well.

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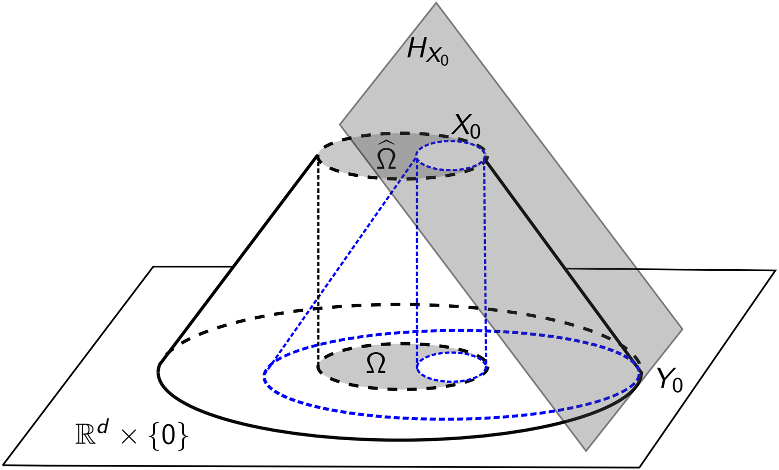
Work out the case of **radial solutions** (when Ω is a ball) by hand.



A scematic view for the homogeneous case.

Convex polygonal domains

- Let $\Omega \subset \mathbb{R}^d \times \{0\}$ be a **convex polygon**, and let $F_1, ..., F_n$ be the **facets** of $\widehat{\Omega} := \Omega \times \{h_0\}$.

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- Then $u(x) = \inf_{1 \leq i \leq n} H_i(x)$, $x \in \mathbb{R}^d$, solves the homogeneous problem.
- The most delicate part is to show that there is no $X \in \mathbb{R}^{d+1}$ in the strip $0 < x_{d+1} < h_0$ where **more** than d planes meet, i.e. the graph of u has **NO** vertex (a geometric proof).

Approximation by polygons: existence

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- Approximate Ω by polygonal domains, and for each polygon take the solution constructed above. Then, the limit will converge to h_* and will give a weak solution for the homogeneous problem (uses the weak* continuity of MA measure).

Proposition (Line segments on the graph)

Let u be any weak solution, and assume X_0 is on the graph of u. Then, there is a line segment though X_0 joining the free boundary with $\mathbb{R}^d \times \{h_0\}$ and lying entirely on the graph of u.

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Every weak solution is a ruled surface

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- The case when $X_0 \in \partial\widehat{\Omega} \cup \Gamma_u$ follows by approximation. □

Comparison principle

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Proof. Argue by contradiction, and use the existence of line segments on the graphs. $\square$.

Proposition

Let Ω be bounded convex $C^{1,1}$-regular domain, and let $h_*(x) = \inf_{X_0 \in \partial \widehat{\Omega}} H_{X_0}(x)$, $x \in \mathbb{R}^d$. Then, Γ_{h_*} is $C^{1,1}$ and h_* is $C^{1,1}$ in $\mathbb{R}^d \setminus \overline{\Omega}$.

Regularity of a weak solution and free boundary

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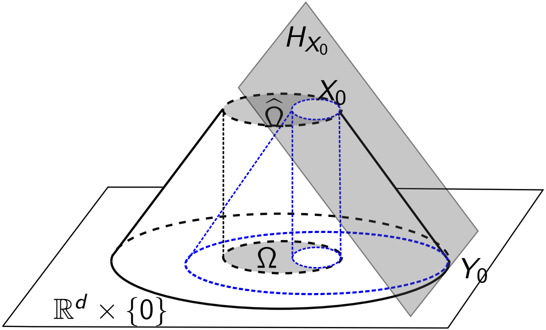
The proof: follow the shared line segment.



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- Strict convexity of h_* follows by comparison with conical solutions.
  A **quantitative version** of strict convexity follows from Blaschke inclusion principle and comparison of the solution with conical barriers (from above).

Elliptic case, $K_0 > 0$, the strategy

- (The class of super-solutions) concave functions $u \in \mathbb{W}_+(K_0, \lambda_0, \Omega)$ s.t. $u = h_0$ on $\partial\Omega$ and

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- (For smoothness of the free boundary) extend the solution beyond the free boundary, to reduce the matters to interior case.

Construction of super-solution

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    - $|\nabla P| \leq \lambda_0$ on $\partial\{P > 0\}$.
- Do this for a dense set of points, and take the infimum: gives an element of $\mathbb{W}_+$.

- Any element of $\mathbb{W}_+$ is larger than the solution to the homogeneous equation. Hence,

$$u_*(x) := \inf_{w \in \mathbb{W}_+} w(x), \;\; x \in \mathbb{R}^d,$$

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Perron in action

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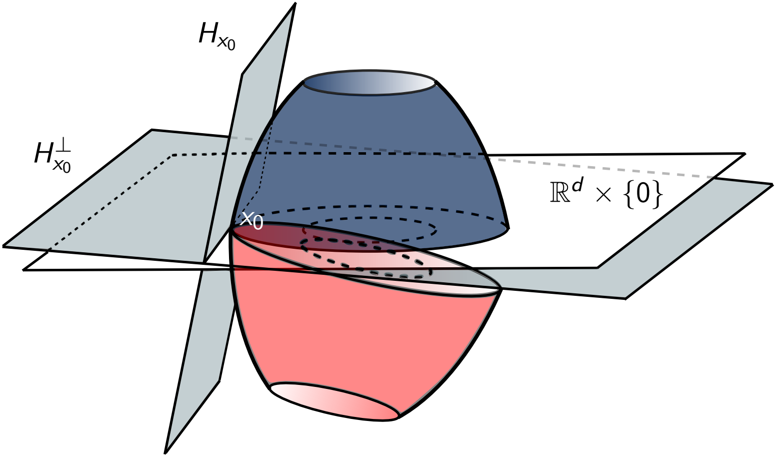
- Solving the Dirichlet problem for affine boundary data, and using strong comparison principle, show u_* solves the equation in $\{u_* > 0\} \setminus \Omega$.
- Still need to show that $|\nabla u_*| = \lambda_0$ on the free boundary (we have only $\leq$ everywhere by construction).

Blaschke extension and the free boundary condition



Reflection of a surface at a single point on the free boundary.

...details

- Define a convex body $\mathcal{S}_*^+$ bounded by the graph(u_*) if $0 < x_{d+1} < h_0$, $\Omega \times \{h_0\}$ if $x_{d+1} = h_0$, and when $x_{d+1} < h_0$ take the intersection of all extreme halfspaces at Γ_{u_*}.

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- For each $x \in \Gamma_{u_*}$, if H_x is an extreme supporting hyperplane to the graph, define $H_x^\perp$ passing through $H_x \cap (\mathbb{R}^d \times \{0\})$ and the normal to H_x.

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- define $\mathcal{S}_x^-$ as the mirror reflection of $\mathcal{S}_x^+$ with respect to $H_x^\perp$.
- Fix $x_0 \in \Gamma_{u_*}$, and take a dense sequence $x_j \subset \Gamma_{u_*}$ near x_0. Define a nested sequence of convex bodies

$$\mathcal{S}^m = S_*^+ \cap \bigcap_{j=1}^m S_{x_j}^-,$$

and take a limit as $m \to \infty$. Call the limit convex body $\mathcal{S}_B$ the Blaschke reflection body.

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- Slightly translate H parallel towards Ω, to H_δ, and in a slab between H and H_δ replace the boundary of $\mathcal{S}_B$ by an exact solution.

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The conclusion is that $|\nabla u_*| = \lambda_0$ everywhere on Γ_{u_*} for the minimal solution, and the free boundary is C^1.

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- We can thus do the same construction with the free boundary as our initial domain.
- Extending in this way, we get that the gradient of extension agrees with the gradient on u_* on the free boundary, and we get a solution across the free boundary. This makes, Γ_{u_*} a level surface of a smooth strictly convex solution, and hence the smoothness of free boundary.

Thank you!