NONLINEAR APPROXIMATION BY RENORMALIZED TRIGONOMETRIC SYSTEM

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ABSTRACT. We study the convergence of greedy algorithm with regard to renormalized trigonometric system. Necessary and sufficient conditions are found for system's normalization to guarantee almost everywhere convergence, and convergence in $L^p(\mathbb{T})$ for $1 < p < \infty$ of the greedy algorithm. Also the non existence is proved for normalization which guarantees convergence almost everywhere for functions from $L^1(\mathbb{T})$, or uniform convergence for continuous functions.

1. INTRODUCTION

Let $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ be a basis in Banach space X satisfying $\inf_n ||\varphi_n||_X > 0$. Then each $f \in$ X can be uniquely decomposed into series with respect to the system Φ, which converges to f in the norm of X:

$$f = \sum_{n=0}^{\infty} c_n(f)\varphi_n,$$

where $c_n(f)$ are the coefficients of the expansion, and $\lim_{n\to\infty} c_n(f) = 0$.

Denote by Λ_N a set consisting of N indices and satisfying

$$\min_{k\in\Lambda_N} |c_k(f)| \geq \max_{k\notin\Lambda_N} |c_k(f)|.$$

Then, the sum

$$\mathrm{G}_N(f) := \mathrm{G}_N(f,\Phi) := \sum_{k\in\Lambda_N} c_k(f)\varphi_k$$

is called N-th *greedy* approximant of f with respect to the system Φ. This method of approximation is called *greedy* algorithm.

A basis Φ is called a *quasi-greedy* basis, if there exists a constant C such that for any $f \in$ X

$$||\mathrm{G}_N(f,\Phi)||_X \leq C||f||_X, \ N = 1, 2... \qquad .$$

P. Wojtaszczyk [23] proved that a basis Φ is quasi-greedy basis if and only if for any $f \in$ X one has

$$\lim_{N\to\infty} ||f - \mathrm{G}_N(f,\Phi)||_X = 0.$$

Convergence of the greedy algorithm for special systems was studied by many authors. T.W. Körner answering a question raised by Carleson and Coifman constructed in [15] a function from $L^2(\mathbb{T})$ and then in [16] a continuous function for which the greedy algorithm by the trigonometric system diverges almost everywhere.

Key words and phrases. non linear approximation, greedy algorithm, trigonometric system.

In [22] V. Temlyakov proved the existence of a function from L^p, $1 \le p < 2$, for which the greedy algorithm with respect to the trigonometric system does not converge in measure, and also existence of a continuous function whose greedy algorithm with respect to the trigonometric system does not converge in L^p, $p > 2$. On the other hand S. Konyagin and V. Temlyakov [13] obtained sufficient conditions for convergence of the greedy algorithm. Similar results concerning convergence and divergence of the greedy algorithm by the Walsh system were obtained by G. Amirkhanyan (see [3]).

M. Grigoryan and A. Sargsyan [5] constructed a continuous function for which the greedy algorithm by the Faber-Schauder system does not converge in measure.

In [18] S. Kostyukovsky and A. Olevskii [18] constructed an orthonormal basis for $L^2(0,1)$ consisting of uniformly bounded functions, such that the greedy algorithm with regard to that system converges almost everywhere for each function from $L^2(0,1)$, and in [19] M. Nielsen constructed an orthonormal system of uniformly bounded functions which is a quasi-greedy basis in $L^p(0,1)$ for all $1 < p < \infty$.

Let $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ be a decreasing sequence of positive numbers. For $f \in \mathrm{X}$ we consider the decreasing rearrangement of absolute values of its non-vanishing coefficients with the weight γ_n:

$$|\gamma_{\sigma(0)} c_{\sigma(0)}(f)| \ge |\gamma_{\sigma(1)} c_{\sigma(1)}(f)| \ge ... |\gamma_{\sigma(n)} c_{\sigma(n)}(f)| \ge ..., \tag{1.1}$$

and define the greedy approximant with the weight Γ as follows:

$$\mathrm{G}_N(f) := \mathrm{G}_N(f,\Phi) := G_N(f,\Phi,\Gamma) := \sum_{n=0}^{N} c_{\sigma(n)}(f)\varphi_{\sigma(n)}, \ N = 1,2,... \quad . \tag{1.2}$$

We denote by $D(f,\Phi,\Gamma)$ the set of permutations σ satisfying (1.1). It is easy to see that (1.2) coincides with the greedy approximant with regard to renormalized system Φ, namely

$$G_N(f,\Phi,\Gamma) = G_N\left(f, \left\{\frac{1}{\gamma_n}\varphi_n\right\}\right), \ N = 1,2,... \quad . \tag{1.3}$$

Greedy algorithm with weight for general systems was considered in [14] and [9]. Weights that guarantee convergence in L^1 of the greedy algorithm with respect to Haar system were characterized in [6]. In [21] convergence almost everywhere of similar algorithm was considered with regard to wavelet systems, and Haar system. Characterization of weighted greedy algorithms which guarantee uniform convergence for continuous functions and convergence almost everywhere for integrable functions for the Haar system is obtained in [17] and [1], and for the classical Franklin system in [2].

In [14] S. Konyagin and V. Temlyakov proved that for any normalized basis Φ in Banach space X, for $\Gamma = \{2^{-n}\}_{n=0}^{\infty}$ and for each $f \in \mathrm{X}$ one has

$$\lim_{N\to\infty} ||f - G_N(f,\Phi,\Gamma)||_{\mathrm{X}} = 0.$$

For the sequence $\Gamma = \{\gamma_n\}$ we denote

$$\omega(\Gamma) = \sup_{m>n\ge 0}\left\{m - n : \frac{\gamma_n}{\gamma_m} \le 2\right\}. \tag{1.4}$$

Example 1.1. *It is clear that* $\omega(\{\alpha^n\}_{n=1}^\infty) < \infty$ *for any positive* $\alpha < 1$, *while* $\omega(\{n^{-1}\}_{n=1}^\infty) = \infty$.

Remark 1.2. *It is easy to see that the condition* $\omega(\Gamma) < \infty$ *is equivalent to the following one*

$$\sup_{m>n} \left\{ m-n: \ \frac{\gamma_n}{\gamma_m} \le \alpha \right\} < \infty, \ \textit{for each } \alpha > 1.$$

Remark 1.3. *Using Remark 1.2 it is easy to see that under the condition* $\omega(\Gamma) = \infty$, *for each* $\alpha > 1$, L *and* S *there exist positive integers* $B > A$ *satisfying*

$$B > A > S, \ B - A > L \ \textit{and} \ \frac{\gamma_A}{\gamma_B} \le \alpha.$$

Denote by $\mathcal{T} = \{e^{inx}\}_{n\in\mathbb{Z}} := \{\psi_n\}_{n=0}^\infty$ the trigonometric system in the following enumeration: $\psi_0 = 1$, $\psi_{2n-1} = e^{inx}$, $\psi_{2n} = e^{-inx}$, for $n = 1, 2, ...$, and let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the unit torus. It is well know that in the mentioned order $\mathcal{T}$ is a basis in $L^p(\mathbb{T})$ for $1 < p < \infty$.

In case of the trigonometric system the rearrangement has the form $\sigma : \mathbb{Z} \to \mathbb{Z}$, i.e. for $f \in L^1(\mathbb{T})$ one considers the sequence

$$|c_0(f)\gamma_0|, \ |c_1(f)\gamma_1|, \ |c_{-1}(f)\gamma_2|, \ |c_2(f)\gamma_3|, \ |c_{-2}(f)\gamma_4|, ...,$$

where $c_n(f)$ are the Fourier coefficients of f, and σ rearranges it in decreasing order.

The present paper aims to study convergence by norm or a.e. for the greedy algorithm with weight with respect to the trigonometric system. But at first we prove the following

Theorem 1.4. *Let* Φ *be a normalized basis in Banach space* X *and assume that the sequence* Γ *satisfies* $\omega(\Gamma) < \infty$. *Then, for each* $f \in \mathrm{X}$ *and* $\sigma \in D(f, \Phi, \Gamma)$,

$$\lim_{N\to\infty} ||f - G_N(f, \Phi, \Gamma)||_{\mathrm{X}} = 0.$$

Since $\omega(\{2^{-n}\}_{n=1}^\infty) < \infty$ the aforementioned theorem of S. Konyagin and V. Temlyakov follows from Theorem 1.4, and from Theorem 1.6 proved below follows that in the class of all bases condition $\omega(\Gamma) < \infty$ is final.

For the trigonometric system we prove the following theorems.

Theorem 1.5. *Let* $\omega(\Gamma) < \infty$ *and* $1 < p < \infty$. *Then for any* $f \in L^p(\mathbb{T})$, *and* $\sigma \in D(f, \mathcal{T}, \Gamma)$ *one has*

1) $\lim_{N\to\infty} ||f - G_N(f, \mathcal{T}, \Gamma)||_p = 0$,
2) $\lim_{N\to\infty} G_N(f, \mathcal{T}, \Gamma)(x) = f(x)$ *a.e.* $x \in \mathbb{T}$.

Theorem 1.6. *Let* $\omega(\Gamma) = \infty$. *Then*

1) *there exists a function* $f \in \bigcap_{1\le p<2} L^p(\mathbb{T})$ *for which the sequence* $\{G_N(f, \mathcal{T}, \Gamma)\}$ *diverges in measure for any* $\sigma \in D(f, \mathcal{T}, \Gamma)$,
2) *there exists a function* $f \in \mathrm{C}(\mathbb{T})$ *for which the sequence* $\{G_N(f, \mathcal{T}, \Gamma)\}$ *diverges in* $||\cdot||_p$ *norm, for any* $2 < p < \infty$ *and each* $\sigma \in D(f, \mathcal{T}, \Gamma)$.

For continuous functions the following holds true.

Theorem 1.7. *Let the sequence* Γ *be fixed. Then*

1) *if* $\omega(\Gamma) = \infty$ *there exists* $f \in C(\mathbb{T})$, *such that the sequence* $\{G_N(f, \mathcal{T}, \Gamma)\}$ *diverges unboundedly a.e. for each* $\sigma \in D(f, \mathcal{T}, \Gamma)$,
2) *if* $\omega(\Gamma) < \infty$ *there exists* $f \in \mathrm{C}(\mathbb{T})$, *such that the sequence* $\{G_N(f, \mathcal{T}, \Gamma)\}$ *diverges at some point for each* $\sigma \in D(f, \mathcal{T}, \Gamma)$.

In the next theorem we study convergence almost everywhere of the greedy algorithm of functions from $L^1(\mathbb{T})$.

Theorem 1.8. *For any sequence* Γ *there exists a function* $f \in L^1(\mathbb{T})$, *such that the sequence* $\{G_N(f, \mathcal{T}, \Gamma)\}$ *diverges unboundedly almost everywhere for any* $\sigma \in D(f, \mathcal{T}, \Gamma)$.

2. Proofs of Theorems 1 – 4

In the proofs below we denote by C absolute constants, which can be different in different formulas. For $f \in L^1(\mathbb{T})$ by sp$f$ denote the spectrum of the Fourier series of f with regard to the trigonometric system, namely sp$f = \{n \in \mathbb{Z} : \ c_n(f) \neq 0\}$.

Proof of Theorem 1. Let $f \in \mathrm{X}$ and a number $\varepsilon > 0$ be fixed. Denote

$$T_\varepsilon(f) := \sum_{n:\ |c_n(f)\gamma_n| > \varepsilon} c_n(f)\varphi_n$$

and

$$N(\varepsilon) = \min\{N \in \mathbb{Z}_+ : \ |c_n(f)\gamma_n| \leq \varepsilon, \ \forall n \geq N\}.$$

Then

$$\{n \in \mathbb{Z}_+ : \ |c_n(f)\gamma_n| > \varepsilon\} \subset \{0, 1, ..., N(\varepsilon)\}, \tag{2.1}$$

and

$$\frac{\varepsilon}{\gamma_{N(\varepsilon)}} \leq |c_{N(\varepsilon)}(f)| \to 0, \ \text{ as } \varepsilon \to 0. \tag{2.2}$$

Now we estimate the following difference

$$||S_{N(\varepsilon)}(f) - T_\varepsilon(f)||_{\mathrm{X}} = \left|\left| \sum_{n \leq N(\varepsilon), \ |c_n(f)\gamma_n| \leq \varepsilon} c_n(f)\varphi_n \right|\right|_{\mathrm{X}} \leq \varepsilon \sum_{n=0}^{N(\varepsilon)} \frac{1}{\gamma_n}. \tag{2.3}$$

Denote $H := \omega(\Gamma) + 1$, then from definition (1.4) and condition $\omega(\Gamma) < \infty$, we have

$$\frac{\gamma_n}{\gamma_m} > 2, \ \text{ where } m - n \geq H.$$

Hence

$$\sum_{n=0}^{N(\varepsilon)} \frac{1}{\gamma_n} = \sum_{r=0}^{H-1} \sum_{n \equiv r (mod \ H)} \frac{1}{\gamma_n} \leq$$

$$\sum_{r=0}^{H-1} \left(\frac{1}{2^{i_r}} + \frac{1}{2^{i_r - 1}} + ... + 1 \right) \frac{1}{\gamma_{N(\varepsilon)}} \leq C \frac{1}{\gamma_{N(\varepsilon)}}, \tag{2.4}$$

where i_r, $r = 0, 1, ..., H-1$ are some indices. From (2.3), (2.4) and (2.2) follows, that

$$\lim_{\varepsilon\to 0}||S_{N(\varepsilon)}(f)-T_\varepsilon(f)||_{\mathrm{X}}=0. \tag{2.5}$$

If the set $D(f,\Phi,\Gamma)$ consists of a single element, then for each $N\in\mathbb{N}$ there exists a number $\varepsilon=\varepsilon(N)>0$ satisfying $G_N(f)\equiv T_\varepsilon(f)$ and hence $\lim\limits_{N\to\infty}||f-G_N(f)||_{\mathrm{X}}=0$.

In case of $\#D(f,\Phi,\Gamma)>1$ denote

(2.6) $\Omega_0=\emptyset,\ \Omega_n=\{k\in\mathbb{Z}_+\setminus(\Omega_1\cup...\cup\Omega_{n-1}):\ |\gamma_k c_k(f)|=|\gamma_n c_n(f)|\},\ n=1,2,...,$

and if $\Omega_n\neq\emptyset$ set $\omega_n=\max\Omega_n$. Now, if $\#\Omega_n>1$, by the same methods as in the proof of (2.4) we get

$$\sum_{k\in\Omega_n}||c_k(f)\varphi_k||_{\mathrm{X}}=|\gamma_{\omega_n}c_{\omega_n}(f)|\sum_{k\in\Omega_n}\frac{1}{\gamma_k}\le$$

$$|\gamma_{\omega_n}c_{\omega_n}(f)|\sum_{k=0}^{\omega_n}\frac{1}{\gamma_k}\le C|c_{\omega_n}(f)|, \tag{2.7}$$

which together with (2.5) proves the theorem.

Theorem 1 is proved.

Proof of Theorem 2. As can be seen from the proof of Theorem 1.4 (see (2.3) and (2.7)), as $\omega(\Gamma)<\infty$ the difference between approximants $G_N(f,\mathcal{T},\Gamma)$, $N=1,2,...$ of the function $f\in L^1(\mathbb{T})$ and some subsequence of the partial sums its Fourier series tends to zero uniformly. Hence the first assertion of the theorem follows from the basis property of the trigonometric system in $L^p(\mathbb{T})$, $1<p<\infty$, and the second one from the celebrated theorem of Carleson-Hunt (see [7]).

Theorem 2 is proved.

Let

$$D_N(x)=\sum_{n=-N}^{N}e^{inx},\ N=0,1,2,...$$

be the Dirichlet kernel for the trigonometric system, and

$$V_N(x)=\frac{1}{N}\sum_{n=N+1}^{2N}D_n(x),\ N=1,2,...$$

be the de la Vallúe Poussin kernel. The following lemma holds true:

Lemma 2.1. *For each $N\in\mathbb{N}$ and $1\le p\le\infty$ one has*

$$||V_N||_p\le CN^{1-1/p}.$$

Proof. As is known $||V_N||_1\le C$ (see [8], p. 125), and also it is easy to see that $||V_N||_2\le CN^{1/2}$ and $||V_N||_\infty\le CN$ for each $N\ge 1$.

For $f\in L^\infty(\mathbb{T})$ from the Hölder inequality we get

$$||f||_p\le ||f||_q^\lambda ||f||_r^{1-\lambda}, \tag{2.8}$$

where

$$1 \le p, q, r \le \infty, \ \frac{\lambda}{q} + \frac{1-\lambda}{r} = \frac{1}{p} \quad \lambda \in [0,1].$$

Now, by using the mentioned estimates for Vallée-Poussin kernel and inequality (2.8) for $(p,q,r) = (p,1,2)$ in case of $1 < p < 2$, and $(p,q,r) = (p,2,\infty)$ in case of $2 < p < \infty$ we obtain the desired estimate.

Lemma is proved. □

Consider majorants of approximants (1.2) for the system $\mathfrak{T}$:

$$G_N^*(f,\sigma,x) := G_N^*(f,x) := \max_{0 \le n \le N} \left| \sum_{k=-n}^{n} c_{\sigma(k)}(f) e^{i\sigma(k)x} \right|, \ x \in \mathbb{T}, \ N = 1,2,..., \tag{2.9}$$

and if P is a trigonometric polynomial denote

$$G^*(P,\sigma,x) := G^*(P,x) := \max_{n \ge 0} \left| \sum_{k=-n}^{n} c_{\sigma(k)}(P) e^{i\sigma(k)x} \right|, \ x \in \mathbb{T}. \tag{2.10}$$

We will need the following lemma from the remarkable work of T. Körner, (see [16], Lemma 7), which we will paraphrase in a slightly different form.

Lemma 2.2. *For each $\varepsilon > 0$ and any $K > 1$ there exist a trigonometric polynomial P, and a measurable set $E \subset \mathbb{T}$ such that*

(i) $||P||_\infty \le \varepsilon$,

(ii) $G^*(P,\sigma,x) > K, \ \ x \notin E$, *for any permutation* $\sigma \in D(P, \mathfrak{T}, \{1\}_{n \in \mathbb{Z}})$,

(iii) $\mu(E) \le \varepsilon$,

(iv) *modules of all non zero coefficients of P are different and if* $|c_n(P)| > |c_m(P)| > 0$ *then* $|c_n(P)| > 2|c_m(P)|$.

Proof. Assertions (i) – (iii) correspond to items (i) – (iii) of Lemma 7 from [16]. The asserion (iv) follows from the proof of Lemma 14 of mentioned paper [16]. □

Proof of Theorem 3. For the proof of the theorem we use the method from [22].

1. According to Theorem 11, chapter 3, [8], there exist polynomials (polynomials of Rudin-Shapiro)

$$R_N(x) = \sum_{n=0}^{N} \varepsilon_n e^{inx}, \quad x \in \mathbb{T}, \quad \varepsilon_n = \pm 1, \qquad N = 1,2,\dots,$$

satisfying

$$||R_N||_\infty \le 5N^{1/2}, \qquad N = 1,2,... \tag{2.11}$$

Since the trigonometric system is orthonormal, if follows from (2.11) that

$$||R_N||_1 \ge CN^{1/2}, \qquad N = 1,2,..., \tag{2.12}$$

where C is an absolute positive constant. For $s = \pm 1$ denote

$$\Lambda_s := \Lambda_s(N) := \{k: \ c_k(R_N) = s\},$$

and let

$$D_\Lambda := \sum_{k \in \Lambda} e^{ikx}.$$

Then

$$R_N = D_{\Lambda_1} - D_{\Lambda_{-1}}.$$

From (2.11) and (2.12) follows existence of absolute constant $c \in (0,1)$ such that

$$\mu\{x \in \mathbb{T}: \ |R_N(x)| > cN^{1/2}\} > c, \qquad N = 1, 2, ...$$

Hence for one of the values $s = \pm 1$ the following inequality holds true:

$$\mu\{x \in \mathbb{T}: \ |D_{\Lambda_s(N)}(x)| > \frac{c}{2} N^{1/2}\} > c. \tag{2.13}$$

Consider the following trigonometric polynomial:

$$g_k(x) := 2^{-k/2} \left(V_{2^k}(x) + s 2^{-2k} R_{2^k}(x) - 2^{-2k} \sum_{n=-2^k}^{-1} e^{inx} \right), \quad x \in \mathbb{T}, \tag{2.14}$$

where $s = \pm 1$ is chosen to satisfy (2.13). Observe that

$$2^{k/2} g_k = (1 + 2^{-2k}) \sum_{n \in \Lambda_s(2^k)} e^{inx} + \sum_{n \notin \Lambda_s(2^k), \ |n| \le 2^{k+1}} \alpha_n e^{inx}, \tag{2.15}$$

where $0 < \alpha_n \le 1 - 2^{-2k}$ for all n. Fix k and let p_0 be an arbitrary positive integer. According to Remark 1.3 choose positive integers $B > A > p_0 + 2^{k+1}$ with the conditions

$$B - A > 2^{k+2} \text{ and } \frac{\gamma_A}{\gamma_B} < \frac{1 + 2^{-2k}}{1 - 2^{-2k}} \tag{2.16}$$

and denote $f_k(x) := e^{i(A + 2^{k+1})x} g_k(x)$. Clearly $\mathrm{sp} f_k \subset [A, B]$. Next, if we set $\gamma_{-1} := \gamma_0$, then in accordance to (2.15), (2.16) and monotonicity of the sequence Γ we get

$$\min_{n \in \Lambda_s(2^k) + A + 2^{k+1}} |c_n(f_k) \gamma_{2n-1}| > \max_{n \notin \Lambda_s(2^k) + A + 2^{k+1}} |c_n(f_k) \gamma_{2n-1}|. \tag{2.17}$$

Also note, that from Lemma 2.1 follows the inequality

$$||f_k||_p \le C 2^{k(1/2 - 1/p)}. \tag{2.18}$$

Now choose increasing sequence of positive integers k_n so that each coefficient in polynomial $g_{k_{n+1}}$ will be smaller than the smallest coefficient of g_{k_n}. Then proceeding from g_{k_n} construct f_{k_n} so that the smallest index of non vanishing coefficients of $f_{k_{n+1}}$ will be bigger than the biggest index of non zero coefficients of f_{k_n}. Set

$$f = \sum_{n=1}^{\infty} f_{k_n}.$$

It follows from (2.18) that $f \in L^p(\mathbb{T})$ for any $p < 2$. Take any permutation $\sigma \in D(f, \Upsilon, \Gamma)$. From the construction of f and (2.17) follows that if we choose positive integers $m_1^{(n)}$ and $m_2^{(n)}$ appropriately then we will get

$$G_{m_1^{(n)}}(f, \Upsilon, \Gamma) - G_{m_2^{(n)}}(f, \Upsilon, \Gamma) = 2^{-k_n/2}(1 + 2^{-2k_n})e^{i(A_n+2^{k_n})x}D_{\Lambda_s(2^{k_n})}.$$

From this, taking into account (2.13) follows that

$$\mu\{x \in \mathbb{T} : \ |G_{m_1^{(n)}}(f, \Upsilon, \Gamma)(x) - G_{m_2^{(n)}}(f, \Upsilon, \Gamma)(x)| > \frac{1}{2}c\} > c,$$

which means divergence in measure of the sequence $\{G_N(f, \Upsilon, \Gamma)\}_{N=1}^{\infty}$.

2. Using the notations of the first part choose $s = \pm 1$ so to get $|\Lambda_s| \geq |\Lambda_{-s}|$ and set

$$g_k := \frac{2^{-k/2}}{k^2}(R_{2^k} + s2^{-k}D_{2^k}). \tag{2.19}$$

As in the first part according to Remark 1.3, for the given number p_0 choose positive integers $B > A > p_0 + 2^k$ so that

$$B - A > 2^{k+1} \text{ and } \frac{\gamma_A}{\gamma_B} < \frac{1 + 2^{-k}}{1 - 2^{-k}}, \tag{2.20}$$

and set $f_k(x) := e^{i(A+2^k)x}g_k(x)$. Then $\mathrm{sp} f_k \subset [A, B]$, and index of the smallest non vanishing coefficient of polynomial f_k will be bigger from the number given in advance. From (2.19), (2.20) and monotonicity of the sequence Γ we get

$$h_k := G_{|\Lambda_s(2^k)|}(f_k, \Upsilon, \Gamma) = \frac{2^{-k/2}}{k^2}(1 + 2^{-k})D_{\Lambda_s(2^k)}. \tag{2.21}$$

Observe that $||h_k||_\infty \geq |h_k(0)| \geq Ck^{-2}2^{k/2}$, hence from the inequality of S. Nikolskii on trigonometric polynomials (see [20], p. 256) we get

$$||h_k||_p \geq C\frac{2^{-k/p}}{k^2}||h_k||_\infty \geq C\frac{1}{k^2}2^{k(1/2-1/p)}. \tag{2.22}$$

Also note that by virtue of (2.11) we have

$$||f_k||_\infty \leq C\frac{1}{k^2}. \tag{2.23}$$

Now, the construction of desired functions from polynomials f_k goes in analogy with part 1. We choose an increasing sequence of positive integers k_n so that each coefficient in the polynomial $g_{k_{n+1}}$ will be bigger than the smallest coefficient in g_{k_n}. Then, starting from g_{k_n} construct f_{k_n} so that the smallest of the indices of non vanishing coefficients of $f_{k_{n+1}}$ will be bigger than the biggest of the indices of non zero coefficients of f_{k_n}. Set

$$f = \sum_{n=1}^{\infty} f_{k_n}.$$

From (2.23) we have $f \in C(\mathbb{T})$. For any permutation $\sigma \in D(f, \Upsilon, \Gamma)$ from the construction of f and (2.21) follows that there exist sequences $m_1^{(n)}$ and $m_2^{(n)}$ for which

$$G_{m_1^{(n)}}(f, \Upsilon, \Gamma) - G_{m_2^{(n)}}(f, \Upsilon, \Gamma) = h_{k_n}.$$

From the latter by virtute of (2.22) follows the assertion of the theorem.
Theorem 3 is proved completely.

Proof of Theorem 4. Assertion 1. We have $\omega(\Gamma) = \infty$. It follows from Lemma 2.2 that there exist trigonometric polynomials P_n and measurable sets $E_n \subset \mathbb{T}$, $n = 1, 2, ...$, satisfying

a) $\sum_{n=1}^{\infty} ||P_n||_\infty < \infty$,

b) $\mu(E_n) \to 0$,

c) $G^*(P_n, \sigma, x) > n,\ x \notin E_n$, for any permutation $\sigma \in D(P_n, \Upsilon, \{1\}_{m\in\mathbb{Z}})$,

d) if $m, k \in \mathrm{sp}P_n$ and $|c_m(P_n)| > |c_k(P_n)|$ then $|c_m(P_n)| > 2|c_k(P_n)|$,

e) $\min_{m\in\mathrm{sp}P_n} |c_m(P_n)| > \max_{m\in\mathrm{sp}P_{n+1}} |c_m(P_{n+1})|$.

According to Remark 1.3 choose positive integers B_n and A_n, $n = 1, 2, ...$ so that

$$A_n < B_n < A_{n+1},\ B_n - A_n > 4|\mathrm{sp}P_n| \text{ and } \frac{\gamma_{A_n}}{\gamma_{B_n}} \le 2, \quad n = 1, 2, ... \tag{2.24}$$

Now set $Q_n(x) = \exp[i(A_n + |\mathrm{sp}P_n|)x]P_n(x)$, $n = 1, 2, ...$ and observe that the spectra of polynomials Q_n are disjoint. We now prove that the function $f := \sum_{n=1}^{\infty} Q_n$ satisfies conditions of the first item of the theorem. Continuity of the function f follows from a). Next, for any permutation $\sigma \in D(f, \Upsilon, \Gamma)$ from condition d), (2.24) and monotonicity Γ follows that

$$\sigma\ |_{\mathrm{sp}Q_n} \in D(Q_n, \Upsilon, \{1\}_{m\in\mathbb{Z}}),\ n = 1, 2, ... \tag{2.25}$$

From e), (2.24), and monotonicity of Γ follows that

$$\min_{k\in\mathrm{sp}Q_n} |c_k(Q_n)\gamma_{2k-1}| > \max_{k\in\mathrm{sp}Q_{n+1}} |c_k(Q_{n+1})\gamma_{2k-1}|,\ n = 1, 2, ... \tag{2.26}$$

From (2.26), (2.25) and condition c) follows that there exists a sequence of indices a_n satisfying

$$G^*_{a_n}(f, \sigma, x) \ge \frac{1}{2}G^*(Q_n, x) \ge \frac{1}{2}n,\ x \notin E_n,$$

from which, taking into account b), follows that the sequence $\{G_N(f, \Upsilon, \Gamma)\}$ diverges almost everywhere, which proves the first part of the theorem.

Assertion 2. D.E. Menshov proved the existence of a continuous function for which any subsequence of partial sums of its Fourier series diverges in at least one point (see [4], p. 327). As was noted in the proof of Theorem 1.5 provided $\omega(\Gamma) < \infty$, for any $f \in L^1(\mathbb{T})$ there is a subsequence of partial sums of its Fourier series so that its difference with $G_N(f, \Upsilon, \Gamma)$, $N = 1, 2, ...$ tends to zero uniformly as N goes to infinity. From the latter it follows that the function from the mentioned work of D. Menshov satisfies the second assertion of the theorem.

Theorem 4 is proved completely.

3. $L^1(\mathbb{T})$ Space. Proof of Theorem 5.

For a function $f \in L^1(\mathbb{T})$ by

$$S_N(f,x) := \sum_{|k|\le N} c_k(f)e^{ikx},\ \ N = 1,2,...$$

we denote the N-th partial sum of its Fourier series. The proof of Theorem 1.8 is carried out by method of Sh. V. Kheladze (see [10], [11]). The next lemma is proved in [10].

Lemma 3.1. *Let*

$$Q_n(x) = \sum_{j=0}^{n-1} \frac{1}{n-j}\cos(n+j)x - \sum_{j=1}^{n} \frac{1}{j}\cos(2n+j)x,\ \ n \in \mathbb{N}.$$

Then

1) $S_{2n-1}(Q_n,x) \ge \frac{1}{2}\ln n$ *for each* $x \in [0, \pi/(6n)]$,
2) $|Q_n(x)| \le 2(\pi+1)$ *for each* $x \in \mathbb{T}$.

The main lemma of the current paragraph is the following:

Lemma 3.2. *Provided* $\omega(\Gamma) < \infty$, *for any* $\varepsilon > 0$, $N \in \mathbb{N}$ *and* $K > 1$ *there exists a real trigonometric polynomial* P *and a measurable set* $E \subset \mathbb{T}$ *such that*

1) $P(x) \ge 0,\ \ x \in \mathbb{T}$,
2) $\int\limits_{\mathbb{T}} P(x)dx = 2\pi$,
3) $c_u(P) = 0,\ 0 < |u| < N$,
4) $G^*(P,\sigma,x) \ge K$, $x \in E$ *for any permutation* $\sigma \in D(P, \mathfrak{T}, \Gamma)$,
5) $\mu(E) \ge 2\pi - \varepsilon$.

Proof. Since $\omega(\Gamma) < \infty$, then for $H := \omega(\Gamma) + 1$ we have $\gamma_n > 2\gamma_m$ for any $m,n \in \mathbb{N}$ provided $m - n \ge H$. Hence for any positive integers $a < b$ we get

$$\gamma_a > 2^l\gamma_b,\ \ \text{where } l = \left[\frac{b-a}{H}\right]. \tag{3.1}$$

Let Q_n be a polynomial defined in Lemma 3.1. For $k = 0,1,2,...$ set

$$Q_{n,k} := \frac{1}{2(\pi+1)}Q_n\left(Hx - \frac{\pi k}{6n}\right),$$

and

$$\Delta_{n,k} := \left[\frac{\pi k}{6nH};\ \frac{\pi(k+1)}{6nH}\right].$$

From polynomials $Q_{n,k}$ we construct a required polynomial by the following way:

$$P(x) := f_n(x) := \prod_{k=0}^{12nH-1} \left[1 + \cos(\lambda_k x)Q_{n,k}(x)\right], \tag{3.2}$$

where positive integers n and λ_k will be chosen later. Immediately note that by virtue of the second item of Lemma 3.1 each polynomial of the form (3.2) satisfies condition 1) of the present lemma. On the other hand it is easy to see that if we choose the sequence λ_k satisfying conditions

$$\lambda_0 > N + 220n^2H^2, \qquad \lambda_k > 3(\lambda_0 + \lambda_1 + ... + \lambda_{k-1}), \quad k = 1, 2, ..., \tag{3.3}$$

then the frequencies of harmonics, which appear after opening the brackets in the product (3.2), will be different from zero, and conditions 2) and 3) of the current lemma will hold true for any polynomial of the form (3.2). In the sequel we will assume that for λ_k the condition (3.3) is satisfied.

At first we consider monomials in the product (3.2), namely polynomials $P_{n,k} := \cos(\lambda_k x)Q_{n,k}(x)$. From the definition of $Q_{n,k}$ one can see that the spectrum of the polynomial $P_{n,k}$ is the set $\{\pm\lambda_k \pm (n+j)H,\ j = 0, 1, ..., n-1\} \cup \{\pm\lambda_k \pm (2n+j)H,\ j = 1, 2, ..., n\}$, and for the corresponding coefficients we have

$$|c_{\pm\lambda_k\pm(n+j)H}(P_{n,k})| = \frac{1}{8(\pi+1)} \cdot \frac{1}{n-j}, \qquad j = 0, 1, ..., n-1 \tag{3.4}$$

and

$$|c_{\pm\lambda_k\pm(2n+j)H}(P_{n,k})| = \frac{1}{8(\pi+1)} \cdot \frac{1}{j}, \qquad j = 1, 2, ..., n. \tag{3.5}$$

Now take $\sigma \in D(P_{n,k}, \mathcal{T}, \Gamma)$. From (3.4), (3.5), (3.1), and monotonicity of the sequence Γ follows that if $u, v \in \mathrm{sp}P_{n,k}$ and $|u| > |v|$, then $\sigma(u) > \sigma(v)$. From the latter, by virtue of condition 2) of Lemma 3.1 we get that

$$G^*(P_{n,k}, \sigma, x) \geq c(\ln n)^{1/2}, \qquad x \in E_{n,k} \cap \Delta_{n,k}, \tag{3.6}$$

where

$$E_{n,k} := \{x \in \mathbb{T} :\ |\cos(\lambda_k x)| \geq (\ln n)^{-1/2}\}.$$

Now, if we take λ_k divisible by $12nH$, then a simple calculation shows that the sets $E_{n,k}$ for any $k = 0, 1, 2, ...$ satisfy

$$\mu(E_{n,k} \cap \Delta_{n,k}) > \mu(\Delta_{n,k}) \left(1 - \frac{2}{\sqrt{\ln n}}\right). \tag{3.7}$$

It is left to prove that by an appropriate choice of λ_k we can make the majorant of f_n close by its value to the majorants of the monomials on the sets $E_{n,k} \cap \Delta_{n,k}$.

Observe that for each $u \in \mathrm{sp}P_{n,k}$ we have $||u| - \lambda_k| \leq 3nH$. After opening the brackets in (3.2) consider polynomials other than monomials general form of which is the following:

$$P_{k_0,k_1,...,k_m} := \cos(\lambda_{k_0}x)\cos(\lambda_{k_1}x) \cdot ... \cdot \cos(\lambda_{k_m}x) Q_{n,k_0}(x) Q_{n,k_1}(x) \cdot ... \cdot Q_{n,k_m}(x), \tag{3.8}$$

where $0 \leq k_0 < k_1 < ... < k_m \leq 12nH - 1$. Clearly if $u \in \mathrm{sp}P_{k_0,k_1,...,k_m}$, then $u = \pm\lambda_{k_0} \pm \lambda_{k_1} \pm ... \pm \lambda_{k_m} + O(1)$, where $O(1)$ is a quantity which by its absolute value does not exceed $36H^2n^2$, as the number of multipliers of $Q_{n,k}$ in the product (3.8) does not exceed $12nH$, and the biggest by absolute value element of a spectrum of each polynomial $Q_{n,k}$ is $3nH$. From (3.3) we get

$$\frac{1}{2}\lambda_{k_{m-1}} \leq ||u| - \lambda_{k_m}| \leq 2\lambda_{k_{m-1}}, \tag{3.9}$$

i.e. each element of the spectrum $P_{k_0,k_1,...,k_m}$ is on the distance of order $\lambda_{k_{m-1}}$ from λ_{k_m}, and the smallest of these distances is greater than $0.5\lambda_0$.

On the other hand it is clear that the smallest modulus of the coefficients of polynomials of the form (3.8) is not less than $a_0 := [8n(\pi+1)]^{-(12nH-1)}$. Using (3.1) we fix λ_0 big enough to get $a_0\gamma_j > \gamma_{j+0.5\lambda_0-3nH}$ for each $j \in \mathbb{N}$. It is easy to see that from the choice of the sequence λ_k and condition (3.9) follows that $\sigma|_{\mathrm{sp}P_{n,k}} \in D(P_{n,k}, \Upsilon, \Gamma)$ and

$$G^*(f_n,\sigma,x) \geq \frac{1}{2}G^*(P_{n,k},\sigma,x), \qquad k = 0,1,...,12nH-1. \tag{3.10}$$

Denote $E_n = \bigcup\limits_{k=0}^{12nH-1}(E_{n,k}\cap\Delta_{n,k})$. From (3.7) we have that

$$\mu(E_n) \geq 2\pi\left(1-\frac{2}{\sqrt{\ln n}}\right), \tag{3.11}$$

and from (3.10), and (3.6) we get

$$G^*(f_n,\sigma,x) \geq c(\ln n)^{1/2}, \qquad x \in E_n. \tag{3.12}$$

From (3.11), and (3.12) follows that if n is sufficiently large, then the constructed polynomial of the form (3.2) will satisfy the lemma.

Lemma is proved. □

Proof of Theorem 5. From Lemma 3.2 follows that there exist a sequence of real trigonometric polynomials P_n, and a sequence of measurable sets $E_n \subset \mathbb{T}$, $n = 1,2,...$ for which

1) $\sum\limits_{n=1}^{\infty}||P_n||_1 < \infty$,
2) $G^*(P_n,\sigma,x) > n$, $x \in E_n$ for any permutation $\sigma \in D(P_n,\Upsilon,\Gamma)$,
3) $\mu(E_n) \to 2\pi$,
4) $(\mathrm{sp}P_n \cap \mathrm{sp}P_m)\setminus\{0\} = \emptyset$, $m \neq n$
5) $\min\limits_{k\in\mathrm{sp}P_n}|c_k(P_n)\gamma_{2|k|+1}| > \max\limits_{k\in\mathrm{sp}P_{n+1}}|c_k(P_{n+1})\gamma_{2|k|}|$.

Set $f = \sum\limits_{n=1}^{\infty}P_n$. From 1) we have that $f \in L^1(\mathbb{T})$. From 4), and 5) follows that $\sigma|_{\mathrm{sp}P_n} \in D(P_n,\Upsilon,\Gamma)$ for each permutation $\sigma \in D(f,\Upsilon,\Gamma)$, and that there exists a sequence a_n for which $G^*_{a_n}(f,\sigma,x) > \frac{1}{2}G^*(P_n,\sigma,x)$. This, by virtue of 2), and 3) implies divergence almost everywhere of the sequence of greedy approximants with weight Γ of the function f.

Theorem 5 is proved.

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