

REGULARITY OF BOUNDARY DATA IN PERIODIC HOMOGENIZATION OF ELLIPTIC SYSTEMS IN LAYERED MEDIA

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ABSTRACT. In this note we study periodic homogenization of Dirichlet problem for divergence type elliptic systems when both the coefficients and the boundary data are oscillating. One of the key difficulties here is the determination of the fixed boundary data corresponding to the limiting (homogenized) problem. This issue has been addressed in recent papers by D. Gérard-Varet and N. Masmoudi [9], and by C. Prange [18], however, not much is known about the regularity of this fixed data. The main objective of this note is to initiate a study of this problem, and to prove several regularity results in this connection.

1. INTRODUCTION

For a bounded domain $D \subset \mathbb{R}^d$ ($d \geq 2$) consider the following problem

$$(1.1) \quad -\nabla \cdot \left(A \left(\frac{\cdot}{\varepsilon} \right) \nabla u \right) (x) = 0, \quad x \in D,$$

with oscillating Dirichlet data

$$(1.2) \quad u(x) = g \left(x, \frac{x}{\varepsilon} \right), \quad x \in \partial D.$$

Here $\varepsilon > 0$ is a small parameter, $A(x) = (A_{ij}^{\alpha\beta}(x))$ is $\mathbb{R}^{N^2 \times d^2}$ -valued function defined on \mathbb{R}^d , where $1 \leq \alpha, \beta \leq d$, $1 \leq i, j \leq N$, and the boundary data $g(x, y)$ is \mathbb{R}^N -valued function defined on $\partial D \times \mathbb{R}^d$. The action of the operator in (1.1) on a vector-function $u = (u_1, \dots, u_N)$ is defined as

$$-(\mathcal{L}_\varepsilon u)_i(x) := \left[\nabla \cdot \left(A \left(\frac{\cdot}{\varepsilon} \right) \nabla u \right) \right]_i(x) = \frac{\partial}{\partial x_\alpha} \left[A_{ij}^{\alpha\beta} \left(\frac{\cdot}{\varepsilon} \right) \frac{\partial u_j}{\partial x_\beta} \right] (x),$$

where $1 \leq i \leq N$. Here and throughout the text, if not stated otherwise, we use the summation convention for repeated indices.

Assumptions. Here we collect all assumptions which will be used when studying problem (1.1)-(1.2).

(A1) (Periodicity) The coefficient tensor A and the boundary data g in its second (oscillating) variable are \mathbb{Z}^d -periodic, that is $\forall y \in \mathbb{R}^d$, $\forall h \in \mathbb{Z}^d$ and $\forall x \in \partial D$ one has

$$A(y + h) = A(y), \quad g(x, y + h) = g(x, y).$$

(A2) (Ellipticity) Coefficients are uniformly elliptic and bounded, that is there exist constants $\Lambda, \lambda > 0$ such that

$$\lambda \xi_\alpha^i \xi_\alpha^i \leq A_{ij}^{\alpha\beta}(x) \xi_\alpha^i \xi_\beta^j \leq \Lambda \xi_\alpha^i \xi_\alpha^i, \quad \forall x \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^{d \times N}.$$

(A3) (Smoothness) We suppose that the boundary data g in both variables, all elements of A , and the boundary of D are infinitely smooth.

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- (A4) (Geometry of the domain) D is a strictly convex domain, i.e. the all principal curvatures of ∂D are bounded away from zero.
- (A5) (Layered medium structure) We assume that the coefficient tensor A is independent of some fixed rational direction, i.e. there exists a non-zero vector $\nu_0 \in \mathbb{Z}^d$ such that $(\nu_0 \cdot \nabla)A(y) = 0$ for all $y \in \mathbb{T}^d$.

The last hypothesis (A5) models media with layered structure, for instance, (A5) includes the class of first order *laminates*. Although homogenization results concerning laminates have been studied in theory, and have independent interest (see e.g. [17]), here the assumption (A5) is technical and is due to our proof.

For each $\varepsilon > 0$ let u_ε be the solution to problem (1.1)-(1.2). Also, for the family of operators $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ let \mathcal{L}_0 be the homogenized (effective) operator in a usual sense of the theory of homogenization (see e.g. [4]). The following homogenization result for u_ε is due to D. Gérard-Varet, and N. Masmoudi.

Theorem 1.1. (see [9, Theorem 1.1]) *Under assumptions (A1)-(A4) there exists a fixed boundary data¹ $g^* \in L^\infty(\partial D)$ such that if u_0 solves*

$$\mathcal{L}_0 u_0(x) = 0, \quad x \in D \quad \text{and} \quad u_0(x) = g^*(x), \quad x \in \partial D,$$

then

$$\|u_\varepsilon - u_0\|_{L^2(D)} \leq C_\alpha \varepsilon^\alpha, \quad \forall \alpha \in \left(0, \frac{d-1}{3d+5}\right).$$

A result related to Theorem 1.1 was proved in our recent work [2] in collaboration with H. Shahgholian, and P. Sjölin, by an approach different than that of [9]. Define projections $P_\gamma^k(x) = x_\gamma(0, \dots, 1, 0, \dots) \in \mathbb{R}^N$ with 1 in the k -th position, where $1 \leq \gamma \leq d$ and $1 \leq k \leq N$. Also, let $\mathcal{L}_\varepsilon^*$ be the adjoint operator to \mathcal{L}_ε , that is the coefficients of $\mathcal{L}_\varepsilon^*$ are set as $(A^*)_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. We then have the following result.

Theorem 1.2. (see [2, Theorem 1.7]) *In the same setting as in Theorem 1.1, assume in addition that $d \geq 3$ and $\mathcal{L}_\varepsilon^*(P_\gamma^k) = 0$ in D for all $1 \leq k \leq N$, $1 \leq \gamma \leq d$, and any $\varepsilon > 0$. Then there exists a function g^* infinitely smooth on ∂D , so that if u_ε is the solution to (1.1)-(1.2) and u_0 of that with homogenized operator \mathcal{L}_0 and boundary data g^* then*

$$\|u_\varepsilon - u_0\|_{L^p(D)} \leq C_p [\varepsilon (\ln(1/\varepsilon))^2]^{1/p},$$

for any $1 \leq p < \infty$. Moreover, g^* may be represented explicitly in terms of the vector field of normals of ∂D , boundary data g , the coefficient tensor A and coefficients of the operator \mathcal{L}_0 .

Using the periodicity condition on the coefficients A one may simplify the condition of Theorem 1.2 on P_γ^k -s. Namely, denote $v_{k,i}^\gamma(x) := (A_{ki}^{\gamma 1}, \dots, A_{ki}^{\gamma d})(x)$, for $x \in \mathbb{R}^d$, $1 \leq k, i \leq N$, $1 \leq \gamma \leq d$, then it is easy to see that the condition $\mathcal{L}_\varepsilon^*(P_\gamma^k) \equiv 0$ is equivalent to

$$(1.3) \quad \operatorname{div}(v_{k,i}^\gamma)(x) = 0, \quad x \in \mathbb{R}^d, \quad 1 \leq k, i \leq N, \quad 1 \leq \gamma \leq d.$$

In the case of $N = 1$ (scalar equations) the last condition means that the rows of the matrix A considered as vector fields in \mathbb{R}^d must be divergence free. The result concerning regularity of g^* contained in Theorem 1.2, although restrictive in terms of the structure of the operator \mathcal{L}_ε , shows that in some cases one may have smooth boundary data for the homogenized problem. Looking ahead let us remark here, that among other things we will

¹This theorem is formulated in [9] with $g^* \in L^p(\partial D)$ for all finite p . However [9] contains a proof of the stronger statement $g^* \in L^\infty(\partial D)$, which we use in the current formulation (in [9] see Proposition 2.4, and the discussion at the end of page 159).

recover this result for g^* (see subsection 4.5) by a different method which will show the smoothness of g^* under conditions of Theorem 1.2 in dimension two as well.

Departing from here, we aim at understanding the regularity of the fixed boundary data g^* defined by Theorem 1.1. Let us first recall some known facts from [9] concerning g^* . For a unit vector $n \in \mathbb{S}^{d-1}$ let P_{n^\perp} be the operator of orthogonal projection on the hyperplane orthogonal to n . Fix $l > 0$ so that $(d-1)l > 1$ and for $\kappa > 0$ set

$$(1.4) \quad \mathcal{A}_\kappa = \{n \in \mathbb{S}^{d-1} : |P_{n^\perp}(\xi)| \geq \kappa |\xi|^{-l} \text{ for all } \xi \in \mathbb{Z}^d \setminus \{0\}\}.$$

A vector $n \in \mathbb{S}^{d-1}$ is called *Diophantine*, if $n \in \mathcal{A}_\kappa$ for some $\kappa > 0$. For $x \in \partial D$ let $n(x)$ be the unit inward normal at x , and define $\Gamma_\kappa = \{x \in \partial D : n(x) \in \mathcal{A}_\kappa\}$. One can see from the analysis of [9] that for any $\kappa > 0$ the restriction of g^* on Γ_κ is Lipschitz continuous with the Lipschitz constant bounded by $C\kappa^{-2}$, where the constant $C = C(A, D, g, d)$. It is shown in [9] that $\sigma(\mathbb{S}^{d-1} \setminus \mathcal{A}_\kappa) \leq C\kappa^{d-1}$, where σ denotes the Lebesgue measure on the unit sphere of \mathbb{R}^d . Also, it is not hard to see that the complement $\mathcal{A}_\kappa^c = \mathbb{S}^{d-1} \setminus \mathcal{A}_\kappa$, while a set of small measure, is everywhere dense and is an open subset of the unit sphere. Next, due to strict convexity of D and smoothness of ∂D , we have that the Gauss map of ∂D , namely $\partial D \ni x \mapsto n(x) \in \mathbb{S}^{d-1}$ is a diffeomorphism, which implies that the sets Γ_κ have similar properties as \mathcal{A}_κ , in particular, the surface measure of Γ_κ decays as $\kappa \rightarrow 0$, and the complement of each Γ_κ is open and dense in ∂D . We see that as $\kappa \rightarrow 0$, the sets Γ_κ cover the entire boundary of D up to measure zero, and hence g^* is defined almost everywhere on ∂D . However, since the upper bound for Lipschitz constant of g^* on Γ_κ , which is $C\kappa^{-2}$, blows up as $\kappa \rightarrow 0$, we *cannot* conclude that there exists an extension of g^* to ∂D which will be continuous at least at a single point. As we will see here, the behaviour of g^* is more regular for layered structures.

For a given domain D with smooth boundary, and $\tau > 0$ set

$$\partial D_\tau = \{x \in \partial D : n(x) \notin \mathbb{R}\mathbb{Q}^d \text{ and } |n(x) \cdot \nu_0| > \tau\},$$

where ν_0 is fixed from assumption (A5). We have the following result.

Theorem 1.3. (The Regularity Theorem) *Let assumptions (A1)-(A5) be in force, and let g^* be defined by Theorem 1.1. Then, for any $\tau > 0$ there exists a constant $C_\tau = C(A, D, g, d, \tau)$ such that*

$$|g^*(x) - g^*(y)| \leq C_\tau |x - y|, \quad \forall x, y \in \partial D_\tau.$$

Corollary 1.4. *g^* has a unique continuous extension to $\{x \in \partial D : n(x) \cdot \nu_0 \neq 0\}$.*

Proof. Note that by Theorem 1.1 g^* is defined almost everywhere on ∂D and we need to extend g^* on a measure zero set of ∂D . By Theorem 1.3 for any $\tau > 0$ the function g^* is uniformly continuous on ∂D_τ , and hence admits a unique continuous extension to $\{x \in \partial D : |n(x) \cdot \nu_0| > \tau\}$. The proof follows by taking $\tau \rightarrow 0$. \square

The next two examples are meant to point out some scenarios when Theorem 1.3 can be used more effectively.

Example 1.5. *Under (A1)-(A4) assume in addition that the coefficient tensor A is independent of the first k coordinates for some $1 \leq k \leq d$, or equivalently that (A5) is satisfied for vectors $\{e_i\}_{i=1}^k$ where $e_i \in \mathbb{R}^d$ is the i -th vector of the standard basis of \mathbb{R}^d . Then, for each $1 \leq i \leq k$ taking e_i as the vector in assumption (A5), and applying Theorem 1.3 k -times, we get that for any $\tau > 0$ there exists a constant $C_\tau = C(A, D, g, d, \tau)$ such that*

$$|g^*(x) - g^*(y)| \leq C_\tau |x - y|, \quad \forall x, y \in \partial D_\tau^{(k)},$$

where

$$\partial D_\tau^{(k)} = \{x \in \partial D : n(x) \notin \mathbb{R}\mathbb{Q}^d \text{ and } \max_{1 \leq i \leq k} |n(x) \cdot e_i| > \tau\}.$$

Likewise, Corollary 1.4 implies that g^* has a unique continuous extension to $\{x \in \partial D : \max_{1 \leq i \leq k} |n(x) \cdot e_i| \neq 0\}$. This shows that for $1 \leq k \leq d-1$ the set of discontinuity of g^* can have Hausdorff dimension at most $d-k-1$, while in the case of $k=d$, i.e. when the coefficients are constant, one gets that g^* extends continuously on the entire boundary of D . The latter statement matches (in a weaker form) with already known result from [2] where it is proved that for constant coefficient operators, the homogenized boundary data is the average of g in its periodic variable, and hence is smooth in particular (see also subsection 4.5).

Example 1.6. Let the domain D be the unit ball of \mathbb{R}^d , and suppose the coefficient tensor $A(x)$ is independent of all variables except possibly variable x_i for some $1 \leq i \leq d$ (i.e. A models a first order laminate). Let also the assumptions (A1)-(A3) be in force. Clearly $\partial D = \mathbb{S}^{d-1}$ and thus g^* is a function on the unit sphere. Then, from Example 1.5 we get that g^* has a unique continuous extension to the unit sphere, except possibly two poles $(0, \dots, 0, \pm 1, 0, \dots, 0) \in \mathbb{S}^{d-1}$ where the non-zero element is in the i -th coordinate.

The strategy of the proof of Theorem 1.3 will be discussed in the next section. In general, without any structural assumptions on the operator, we do not know whether g^* has an extension to ∂D which is continuous at least at a single point on the boundary. Also, it will be very interesting to see if the regularity of g^* can have some impact on the speed of convergence in the actual homogenization problem (1.1)-(1.2). A positive sign in this direction is Theorem 1.2, although there the smoothness of g^* is a corollary, rather than a starting point.

Notation. We fix some notation and conventions that will be used in the sequel. An integer d always stands for the dimension of \mathbb{R}^d , and throughout the paper we have $d \geq 2$. By $N \in \mathbb{N}$ we denote the number of equations in (1.1).

\mathbb{S}^{d-1} is the unit sphere, and \mathbb{T}^d is the unit torus of \mathbb{R}^d . By $\mathbb{R}\mathbb{Q}^d$ we denote the set of all vectors from \mathbb{R}^d that are scalar multiples of vectors with all entries being rational numbers. We call elements of $\mathbb{R}\mathbb{Q}^d$ *rational* vectors (directions, if they have length one), and the complement of $\mathbb{R}\mathbb{Q}^d$ is referred to as *irrational* vectors (correspondingly directions).

In the sequel notation $d\sigma$ in integrals stands for standard surface measure.

For a vector $n \in \mathbb{S}^{d-1}$ we set $\Omega_n = \{x \in \mathbb{R}^d : x \cdot n > 0\}$, where “ \cdot ” is the usual inner product in \mathbb{R}^d . For $x \in \mathbb{R}^d$, if no confusion arises we let $|x|$ be its Euclidean norm. For $k \in \mathbb{N}$ we denote by $M_k(\mathbb{R})$ the set of $k \times k$ matrices with real entries, and by $O(k)$ the set of $k \times k$ orthogonal matrices.

Throughout the text the letter C with or without a subscript denotes an absolute constant which may vary from formula to formula. For two quantities a and b we write $a \lesssim b$ if there is an absolute constant C such that $a \leq Cb$. For a, b depending on some parameter δ , we may write $a \lesssim_\delta b$ or $a \leq C_\delta b$, to point out that the constant in the inequality depends on δ and is otherwise absolute.

The word “smooth” always means differentiable of class C^∞ .

2. BOUNDARY LAYER SYSTEMS AND CONSTRUCTION OF HOMOGENIZED DATA g^*

For a unit vector $n \in \mathbb{R}^d$ and scalar $a \in \mathbb{R}$ set $\Omega_{n,a} = \{x \in \mathbb{R}^d : x \cdot n > a\}$, and for a smooth and \mathbb{Z}^d -periodic vector-function v_0 consider the following problem

$$(2.1) \quad \begin{cases} -\nabla \cdot A(y) \nabla v(y) = 0, & y \in \Omega_{n,a}, \\ v(y) = v_0(y), & y \in \partial\Omega_{n,a}. \end{cases}$$

Problems of the form (2.1) will be referred to as *boundary layer systems*. These type of systems have a central role in the theory of periodic homogenization of Dirichlet problem

for divergence type elliptic operators with (simultaneously) oscillating coefficients and boundary data. In a nutshell, the relevance of (2.1) to homogenization of (1.1)-(1.2) can be traced as follows. In a small neighbourhood of a given point $x_0 \in \partial D$ having normal n , one tries to attribute oscillations of u_ε caused by boundary data to a new independent variable y which leads to approximating the solution u_ε by a function of the form $v(x, x/\varepsilon)$ periodic in its second (oscillating) variable. Plugging such a v into the equation formally, leads to a problem of the form (2.1) where taking $\varepsilon \rightarrow 0$ amounts to asymptotics of v far away from the boundary of corresponding halfspace Ω_n , which is meant to model the halfspace containing D determined by tangent hyperplane of ∂D at x_0 . Questions concerning well-posedness of boundary layer systems and behaviour of solutions far away from the boundary of the corresponding hyperplane, form a significant portion of the analysis toward obtaining quantitative results for homogenization of the mentioned class of Dirichlet problems. We refer the reader to [9], [10], and [18] for details concerning emergence of boundary layer systems in homogenization and their analysis. We will however, recall the following result which is necessary for our purposes.

Theorem 2.1. (see² [18, Theorem 1.2]) *In (2.1) assume A satisfies conditions (A1)-(A3), $v_0 \in C^\infty(\mathbb{T}^d; \mathbb{R}^N)$, and let $n \in \mathbb{S}^{d-1}$. Then*

1. *there exists a unique solution $v \in C^\infty(\overline{\Omega_{n,a}}) \cap L^\infty(\Omega_{n,a})$ of (2.1) such that*

$$\|\nabla v\|_{L^\infty(\{y \cdot n > t\})} \rightarrow 0, \text{ as } t \rightarrow \infty,$$

$$\int_a^\infty \|(n \cdot \nabla)v\|_{L^\infty(\{y \cdot n - t = 0\})}^2 dt < \infty,$$

2. *if in addition $n \notin \mathbb{R}\mathbb{Q}^d$, then there exists a boundary layer tail $v^\infty \in \mathbb{R}^N$ independent of a so that*

$$v(y) \rightarrow v^\infty, \text{ as } y \cdot n \rightarrow \infty \text{ where } y \in \Omega_{n,a},$$

and the convergence is locally uniform with respect to the tangential variables.

Now, following [9] and [18] we describe the construction of the homogenized boundary data. First, consider the case when boundary data g in (1.2) can be factored into independent components depending on x and y . Namely, assume that there exists a smooth v_0 defined on \mathbb{T}^d with values in $M_N(\mathbb{R})$ and some smooth g_0 defined on ∂D and with values in \mathbb{R}^N so that $g(x, y) = v_0(y)g_0(x)$. Next, take any $x \in \partial D$ such that $n(x) \notin \mathbb{R}\mathbb{Q}^d$, and for $n(x)$ consider the boundary layer system (2.1) with boundary data v_0 . Then let $v^\infty(x)$ be the constant field provided by Theorem³ 2.1. Observe, that we do not need to specify the parameter a in (2.1), since in view of Theorem 2.1 the boundary layer tail v^∞ is independent of a for irrational directions. Thus, without loss of generality we may assume that $a = 0$. Finally, for $x \in \partial D$ satisfying $n(x) \notin \mathbb{R}\mathbb{Q}^d$ set

$$g^*(x) := v^\infty(n(x))g_0(x).$$

As we have discussed above, the Gauss map of ∂D realizes a diffeomorphism between ∂D and \mathbb{S}^{d-1} , hence g^* is defined almost everywhere on ∂D . The general case proceeds

²The current formulation is slightly different from the original one, in that we only require n to be irrational in part 2 of the Theorem. This, however, is the outcome of the original proof, since part 1 shows that the only solution with the mentioned properties is the one given by Poisson kernel, and the only recourse to irrationality of n is necessary for the asymptotic analysis of the solution away from the boundary (cf. paragraphs (1) and (2) in [9, subsection 2.1]).

³It should be remarked that technically Theorem 2.1 is formulated for the case when the boundary data is an N -dimensional vector, while here we need an $N \times N$ matrix. Clearly this is not an issue, since one may treat each column of the matrix separately, as is mentioned e.g. in [9].

by approximation. Using periodicity of g in y and its smoothness we have the following expansion

$$g(x, y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(x) e^{2\pi i y \cdot \xi} =: \sum_{\xi \in \mathbb{Z}^d} g_\xi(x, y),$$

where the series converge uniformly and absolutely. Here $g_\xi(x, y)$ is factored since $c_\xi \in \mathbb{R}^N$ and we may identify the exponential $e^{2\pi i \xi \cdot y}$ with $e^{2\pi i \xi \cdot y} I_N$, where $I_N \in M_N(\mathbb{R})$ is the identity matrix. We let v_ξ^∞ be the constant field corresponding to the ξ -th exponential. Then, it is shown in [9] that the homogenized boundary data is given by

$$(2.2) \quad g^*(x) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(x) v_\xi^\infty(n(x)) =: \sum_{\xi \in \mathbb{Z}^d} g_\xi^*(x),$$

where $x \in \partial D$ and $n(x) \notin \mathbb{R}\mathbb{Q}^d$. We refer the reader to Section 4.2 of [9] for the details⁴.

A starting point of our analysis will be a representation formula for v^∞ computed in [18], for which we need some preliminary definitions. Recall that A^* is the coefficient tensor for the adjoint operator, i.e. $(A^*)_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$. Next, for all $1 \leq \gamma \leq d$ we let $v^{*,\gamma} \in M_N(\mathbb{R})$ be the solution (in the sense of Theorem 2.1) to the following system

$$(2.3) \quad \begin{cases} -\nabla_{\tilde{y}} \cdot A^*(\tilde{y}) \nabla_{\tilde{y}} v^{*,\gamma}(\tilde{y}) = 0, & \tilde{y} \in \Omega_n, \\ v^{*,\gamma}(\tilde{y}) = -\chi^{*,\gamma}(\tilde{y}), & \tilde{y} \in \partial\Omega_n, \end{cases}$$

where $\chi^{*,\gamma} \in M_N(\mathbb{R})$ is the solution to the following *cell-problem*

$$(2.4) \quad \begin{cases} -\nabla_y \cdot A^*(y) \nabla_y \chi^{*,\gamma}(y) = \partial_{y_\alpha} A^{*,\alpha\gamma}, & y \in \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \chi^{*,\gamma}(y) dy = 0. \end{cases}$$

We will also need a certain analogue of the notion of *mean-value* for almost-periodic functions given by the next lemma.

Lemma 2.2. (see [22, Theorem S.3]) *Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be almost-periodic. Then there exists a scalar $\mathcal{M}(f)$ such that for any $\varphi \in L^1(\mathbb{R}^d)$ one has*

$$\int_{\mathbb{R}^d} \varphi(y) f(\lambda y) dy \rightarrow \mathcal{M}(f) \int_{\mathbb{R}^d} \varphi(y) dy, \quad \text{as } \lambda \rightarrow \infty.$$

The following useful formula for $v^\infty(n)$ defined by Theorem 2.1 is due to C. Prange (see formula (6.4) in [18]). Keeping the notation of Theorem 2.1 and Lemma 2.2 we have

$$(2.5) \quad v^\infty(n) = \int_{\partial\Omega_n} \partial_{y_\alpha} G^0(n, y) d\sigma(y) \times \left[\mathcal{M}\{A^{\beta\alpha}(y) v_0(y) n_\beta\} + \right. \\ \left. \mathcal{M}\left\{ \partial_{y_\beta} (\chi^{*,\alpha})^t(y) A^{\beta\gamma}(y) v_0(y) n_\gamma \right\} + \right. \\ \left. \mathcal{M}\left\{ \partial_{y_\beta} (v^{*,\alpha})^t(y) A^{\beta\gamma}(y) v_0(y) n_\gamma \right\} \right],$$

where $\Omega_n = \{x \in \mathbb{R}^d : x \cdot n > 0\}$ and G^0 is the Green's kernel corresponding to the homogenized constant coefficient operator $-\nabla \cdot A^0 \nabla$ in domain Ω_n . Also, the averages $\mathcal{M}\{\cdot\}$ are understood for restrictions of functions on the hyperplane Ω_n , that is one may apply Lemma 2.2 after rotating the hyperplane $\partial\Omega_n$ to $\mathbb{R}^{d-1} \times \{0\}$. More precisely, for $F : \mathbb{R}^d \rightarrow \mathbb{R}^N$ and $n \in \mathbb{S}^{d-1}$ one takes a matrix $M \in O(d)$ such that $M e_d = n$ and applies Lemma 2.2 for a function $f(z') = F(M(z', 0))$, $z' \in \mathbb{R}^{d-1}$. We do not enter into details

⁴In fact [9] only treats Diophantine normals in a sense of (1.4). As we have seen above all points of ∂D up to measure zero satisfy (1.4) for some parameter $\kappa > 0$, and hence (2.2) is defined almost everywhere on ∂D . The extension of (2.2) to all irrational directions follows from Theorem 2.1.

concerning almost-periodic functions, as here our treatment will be self-contained. The interested reader is referred, for example, to [22] for particulars.

The strategy of the proof. We are now in a position to give an outline of the strategy of the proof of Theorem 1.3. From (2.2) and (2.5) it is apparent that the regularity of g^* depends on the regularity of v^∞ with respect to the normal directions, and we will proceed by analysing the dependence on the normal field of the quantities involved in (2.5).

In Section 3 we show, mostly through linear algebra and some basic properties of Green's kernel, that integrated Green's kernel in (2.5) as a matrix-function of n is smooth on \mathbb{S}^{d-1} . It should be noted that we do not prove the smoothness of Green's kernel itself with respect to n . That problem can be analysed using Lemma A.1 which also indicates that there are some topological objections to global smoothness of these kernels on \mathbb{S}^{d-1} . Next, using Fourier-analytic approach (Lemma 2.3 and its corollaries) we show that \mathcal{M} -averages are well-behaved for a class of almost-periodic functions. In particular that allows us to compute the first two averages in (2.5) explicitly. Since in general the corrector $v^{*,\alpha}$ does not fall into the realm of applicability of Lemma 2.3, we analyse the last average of (2.5) in Section 4 - the main part of this paper. It is there that assumption (A5) enters the proof, allowing us to transform the boundary layer system for $v^{*,\alpha}$ from Ω_n to \mathbb{R}_+^d by linear change of variables, while keeping the periodicity of the operator and the boundary data intact (however, by the price of making the ellipticity constant of the operator worse). Then, using Tartar's construction (Theorem 4.2) we show that the solution to (2.3) has exponentially decaying gradient in the normal direction and is periodic in tangential directions. We then use these properties to get expansion of the corrector into series of exponentials (formula (4.40)) and show Lipschitz regularity of coefficients of the expansion with respect to normal directions (Lemma 4.8) by elliptic regularity arguments. This enables us to apply corollaries of Lemma 2.3 to the last average of (2.5) as well. Finally, the proof of Theorem 1.3 glues the analysis for v^∞ with expansion in (2.2) to produce the result for g^* .

We finish this section by two observations. First, we compute the constant \mathcal{M} for some class of almost-periodic functions, and second, we establish a uniform bound on the constant field of Theorem 2.1 in terms of the corresponding boundary data.

Lemma 2.3. *Let T be a fixed $d \times d$ matrix with rational coefficients, and assume we are given a function $f(y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(f) e^{2\pi i T \xi \cdot y}$, $y \in \mathbb{R}^d$, where each $c_\xi \in \mathbb{C}$, and $\sum_{\xi \in \mathbb{Z}^d} |c_\xi(f)| < \infty$. For a unit vector $n \notin \mathbb{R}\mathbb{Q}^d$ and a matrix $M \in O(d)$ satisfying $M e_d = n$, set $h(z') = f(M(z', 0))$, where $z' \in \mathbb{R}^{d-1}$. Then*

$$\mathcal{M}(h) = \sum_{\xi: T\xi=0} c_\xi(f).$$

Proof. To compute $\mathcal{M}(h)$ fix some $\varphi \in C_0^\infty(\mathbb{R}^{d-1})$, set $\phi_\xi(z') = T\xi \cdot M(z', 0)$ for $\xi \in \mathbb{Z}^d$ and consider

$$(2.6) \quad \mathcal{J}_\xi(\lambda) = \int_{\mathbb{R}^{d-1}} \varphi(z') e^{-2\pi i \lambda \phi_\xi(z')} dz', \quad \lambda > 1.$$

The proof will be completed once we show that for each ξ satisfying $T\xi \neq 0$ one has $\mathcal{J}_\xi(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. We henceforth assume that $T\xi \neq 0$.

It follows from the definition of the matrix M that $M = [N|n]$, where N is a $d \times (d-1)$ matrix. We have $T\xi \cdot M(z', 0) = N^t T\xi \cdot z'$, and hence $\nabla' \phi_\xi(z') = N^t T\xi$, for all $z' \in \mathbb{R}^{d-1}$ where ∇' is the gradient in \mathbb{R}^{d-1} . But as M is orthogonal, it preserves the Euclidean

length, consequently

$$|T\xi| = |M^t T\xi| = |(N^t T\xi, n \cdot T\xi)| = |(\nabla' \phi_\xi(z'), n \cdot T\xi)|.$$

Therefore, if we assume that $\nabla' \phi_\xi(z') = 0' \in \mathbb{R}^{d-1}$, we get $|n \cdot T\xi| = |T\xi|$, which, by the equality case in Cauchy-Schwarz inequality infers $n = T\xi/|T\xi|$. Since T has rational entries, it follows that $T\xi \in \mathbb{R}\mathbb{Q}^d$, and hence so is n , contradicting the assumption that n is not rational. We thus conclude that $\nabla' \phi_\xi(z') \neq 0'$. Using this, we invoke integration by parts in (2.6) (cf. “the principle of the non-stationary phase” in [21], p. 341, Prop. 4) and get that $\lim_{\lambda \rightarrow \infty} \mathcal{J}_\xi(\lambda) = 0$, for any $\xi \in \mathbb{Z}^d$ with the property $T\xi \neq 0$, which completes the proof of the lemma. \square

For vector-valued functions, in view of the linearity of the averaging operator \mathcal{M} , and choosing matrix-valued test functions in the proof of Lemma 2.3, we immediately get the following.

Corollary 2.4. *For $k \in \mathbb{N}$ assume $f = (f_1, \dots, f_k)$ where each component f_i satisfies Lemma 2.3. Similarly, define $h = (h_1, \dots, h_k)$. Then*

$$\mathcal{M}(h) = (\mathcal{M}(h_1), \dots, \mathcal{M}(h_k)).$$

Observe that if T in Lemma 2.3 is the identity matrix, then f is \mathbb{Z}^d -periodic, and c_ξ is the ξ -th Fourier coefficient of f . This observation directly implies the independence of the first two averages involved in the formula (2.5) from the normal $n \notin \mathbb{R}\mathbb{Q}^d$. Namely, since A , v_0 , and $\chi^{*,\gamma}$ are all \mathbb{Z}^d -periodic, from Lemma 2.3 and Corollary 2.4 we get

$$(2.7) \quad \mathcal{M}\{A^{\beta\alpha}(y)v_0(y)n_\beta\} = \mathcal{M}\{A^{\beta\alpha}(y)v_0(y)\}n_\beta = c_0(A^{\beta\alpha}v_0)n_\beta,$$

and

$$(2.8) \quad \mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)n_\gamma\right\} = \mathcal{M}\left\{\partial_{y_\beta}(\chi^{*,\alpha})^t(y)A^{\beta\gamma}(y)v_0(y)\right\}n_\gamma = c_0[\partial_{y_\beta}(\chi^{*,\alpha})^t A^{\beta\gamma}v_0]n_\gamma,$$

where we have $n \notin \mathbb{R}\mathbb{Q}^d$, and $c_0(f)$ denotes the 0-th Fourier coefficient of \mathbb{Z}^d -periodic function f , i.e. the integral of f over \mathbb{T}^d . Note that at this stage we are not able to apply Lemma 2.3 to the last average in (2.5).

We will also need a setting when we apply \mathcal{M} on a one-parameter family of functions. The next statement follows from Lemma 2.3 in a straightforward manner.

Corollary 2.5. *Let T , n , and M be as in Lemma 2.3, and let \mathcal{E} be some fixed set of parameters. Suppose for each $\tau \in \mathcal{E}$ we have a function $f_\tau(y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(f_\tau) e^{2\pi i T\xi \cdot y}$, $y \in \mathbb{R}^d$, where each $c_\xi(f_\tau) \in \mathbb{C}$, $\sum_{\xi \in \mathbb{Z}^d} |c_\xi(f_\tau)| < \infty$, and for some absolute constant C_0 one has*

$$|c_\xi(f_\tau) - c_\xi(f_\sigma)| \leq C_0 |\tau - \sigma|, \quad \tau, \sigma \in \mathcal{E} \text{ and } \xi \in \mathbb{Z}^d.$$

Then, for any $g \in C^\infty(\mathbb{T}^d)$, setting $h_\tau(z') = (fg)(M(z', 0))$, where $z' \in \mathbb{R}^{d-1}$, we get

$$|\mathcal{M}(h_\tau) - \mathcal{M}(h_\sigma)| \leq C_g |\sigma - \tau|, \quad \sigma, \tau \in \mathcal{E}.$$

Proof. By $c_\xi(g)$ denote the ξ -th Fourier coefficient of g . Then by Lemma 2.3 we have

$$\mathcal{M}(h_\tau) = \sum_{\xi: T\xi \in \mathbb{Z}^d} c_\xi(f_\tau) c_{-T\xi}(g), \quad \tau \in \mathcal{E}.$$

The proof now follows by writing $|\mathcal{M}(h_\tau) - \mathcal{M}(h_\sigma)| \leq C_g |\sigma - \tau| \sum_{\xi \in \mathbb{Z}^d} |c_\xi(g)|$, where convergence of the series is due to the smoothness of g . \square

Again, generalization to the vector-valued case is trivial. We next proceed to a uniform estimate for the boundary layer tail. The claim of the next lemma follows from the Poisson representation of solutions proved in [18] and a bound for Poisson kernel proved in [9]. Due to the lack of an explicit reference we include the proof here.

Lemma 2.6. *Keeping the assumptions and notation of Theorem 2.1, for a unit vector $n \notin \mathbb{RQ}^d$ and boundary data v_0 let v^∞ be the corresponding constant field. Then there exists a constant $C = C(A, d)$ independent of n and v_0 , such that $|v^\infty| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)}$.*

Proof. By [18, Section 3.2] for the solution of (2.1) one has

$$v(y) = \int_{\partial\Omega_n} P(y, \tilde{y}) v_0(\tilde{y}) d\sigma(\tilde{y}), \quad y \in \Omega_n,$$

where P is the Poisson kernel for (2.1), and satisfies the estimate (see [9, Lemma 2.5])

$$(2.9) \quad |P(y, \tilde{y})| \leq C \frac{y \cdot n}{|y - \tilde{y}|^d},$$

for all $d \geq 2$, $y \in \Omega_n$ and $\tilde{y} \in \partial\Omega_n$, and the constant C depending on the operator and dimension d only. Using (2.9) one gets

$$|v(y)| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)} \int_{\tilde{y} \cdot n = 0} \frac{y \cdot n}{|y - \tilde{y}|^d} d\sigma(\tilde{y}).$$

For $M \in O(d)$ satisfying $n = Me_d$ make a change of variables in the last integral by $y = Mz$ and $\tilde{y} = M\tilde{z}$. Due to orthogonality of M we have $M^t n = e_d$, hence for any $y \in \Omega_n$ we get $y \cdot n = z \cdot M^t n = z \cdot e_d = z_d > 0$ from which it follows that

$$|v(Mz)| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)} z_d \int_{\tilde{z}_d = 0} \frac{d\sigma(\tilde{z})}{|z - \tilde{z}|^d} = C \frac{\|v_0\|_{L^\infty(\mathbb{T}^d)}}{z_d^{d-1}} \int_{\tilde{z}_d = 0} \frac{d\sigma(\tilde{z})}{\left[1 + \sum_{i=1}^{d-1} \left(\frac{z_i - \tilde{z}_i}{z_d}\right)^2\right]^{d/2}}.$$

Setting $\tau_i := (z_i - \tilde{z}_i)/z_d$, $i = 1, 2, \dots, d-1$ in the last integral, we obtain

$$|v(Mz)| \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)} \int_{\mathbb{R}^{d-1}} \frac{d\tau}{(1 + |\tau|^2)^{d/2}} \leq C \|v_0\|_{L^\infty(\mathbb{T}^d)},$$

finishing the proof. \square

3. REGULARITY OF INTEGRATED GREEN'S KERNELS WITH RESPECT TO NORMALS

In this section we study regularity of integrated Green's kernels in formula (2.5) with respect to normals $n \in \mathbb{S}^{d-1}$. We start with some basic preliminaries.

For a coefficient tensor A and a halfspace $\Omega \subset \mathbb{R}^d$, the Green's kernel $G = G(y, \tilde{y}) \in M_N(\mathbb{R})$ corresponding to the operator $-\nabla \cdot A(y)\nabla$ in domain Ω is a matrix-function satisfying the following elliptic system

$$(3.1) \quad \begin{cases} -\nabla_y \cdot A(y)\nabla_y G(y, \tilde{y}) = \delta(y - \tilde{y})I_N, & y \in \Omega, \\ G(y, \tilde{y}) = 0, & y \in \partial\Omega, \end{cases}$$

for any $\tilde{y} \in \Omega$, where δ is the Dirac distribution and $I_N \in M_N(\mathbb{R})$ is the identity matrix. To have a quick reference to this situation, we will say that G is the Green's kernel for the pair (A, Ω) . The existence and uniqueness of Green's kernels for divergence type elliptic systems in halfspaces is proved in [13, Theorem 5.4] for $d \geq 3$, and in [7, Theorem 2.21] for $d = 2$. Moreover, if A^* is the coefficient tensor for the adjoint operator, and G^* is the corresponding Green's kernel, then one has the following symmetry relation

$$(3.2) \quad G^t(y, \tilde{y}) = G^*(\tilde{y}, y), \quad y, \tilde{y} \in \Omega.$$

Let B^0 be a constant coefficient elliptic tensor and $G^0(z, \tilde{z})$ be the Green's kernel for the pair (B^0, \mathbb{R}_+^d) . Fix a unit vector $n \in \mathbb{S}^{d-1}$, along with a matrix $M \in O(d)$ satisfying $Me_d = n$. Note, that we have no assumption on n being a rational or an irrational direction. For $y, \tilde{y} \in \Omega_n$ set $G^n(y, \tilde{y}) := G^0(M^t y, M^t \tilde{y})$, we now determine a system of equations satisfied by the matrix G^n .

Clearly, for any $y \in \partial\Omega_n$ one has $M^t y \in \partial\mathbb{R}_+^d$ and hence $G^n(y, \tilde{y}) = 0$, so we get a zero boundary condition for G^n in Ω_n for any $\tilde{y} \in \Omega_n$. To get the system for G^n , let us rewrite the system in the definition of the Green's kernel in (3.1). Let $G^0 = (G_{kj}^0) \in M_N(\mathbb{R})$, then according to (3.1) for all $1 \leq i, k \leq N$, we have

$$(3.3) \quad -\partial_{z_\alpha}(B_{ij}^{0,\alpha\beta}\partial_{z_\beta}G_{kj}^0(z, \tilde{z})) = \delta(z - \tilde{z})\delta_{ik}, \quad z \in \mathbb{R}_+^d,$$

where δ_{ik} is the Kronecker delta. For fixed $1 \leq i, j \leq N$ denote $B_{ij}^0 := (B_{ij}^{0,\alpha\beta}) \in M_d(\mathbb{R})$, then with this notation (3.3) becomes

$$-\nabla_z \cdot B_{ij}^0 \nabla_z G_{kj}^0(z, \tilde{z}) = \delta(z - \tilde{z})\delta_{ik}.$$

Now, fix $\tilde{y} \in \Omega_n$, then for any $1 \leq \alpha \leq d$ we have

$$\partial_{y_\alpha} G_{kj}^n(y, \tilde{y}) = \partial_{z_1} G_{kj}^0(M^t y, M^t \tilde{y})m_{\alpha 1} + \dots + \partial_{z_d} G_{kj}^0(M^t y, M^t \tilde{y})m_{\alpha d},$$

and hence $\nabla_y G_{kj}^n(y, \tilde{y}) = M \nabla_z G_{kj}^0(M^t y, M^t \tilde{y})$, from which we obtain

$$(3.4) \quad \nabla_y \cdot B_{ij}^0 \nabla_y G_{kj}^n(y, \tilde{y}) = \nabla_z \cdot M^t B_{ij}^0 M \nabla_z G_{kj}^0(z, \tilde{z}),$$

where $z = M^t y$ and $\tilde{z} = M^t \tilde{y}$. Observe that by non-degeneracy of M we have $\delta(z - \tilde{z}) = \delta(M^t(y - \tilde{y})) = \delta(y - \tilde{y})$, which in combination with (3.4) implies the following.

Claim 3.1. *Let $n \in \mathbb{S}^{d-1}$ be any, and $M \in O(d)$ be such that $Me_d = n$. If $G^{0,n}(z, \tilde{z})$ is the Green's kernel for the pair $(M^t B^0 M, \mathbb{R}_+^d)$, then $G^n(y, \tilde{y}) := G^{0,n}(M^t y, M^t \tilde{y})$ is the Green's kernel for the pair (B^0, Ω_n) , where $M^t B^0 M$ is understood in accordance with (3.4).*

Now let $G^n(y, \tilde{y})$ be the Green's kernel for the pair (A^0, Ω_n) , where A^0 is the homogenized tensor corresponding to $A(y)$. For $1 \leq \alpha \leq d$, set

$$(3.5) \quad \mathcal{J}^\alpha(n) = \int_{\partial\Omega_n} \partial_{\tilde{y}_\alpha} G^n(n, \tilde{y}) d\sigma(\tilde{y}),$$

which is precisely the term involved in the formula (2.5). Let us stress that $\mathcal{J}^\alpha(n)$ is well-defined for any $n \in \mathbb{S}^{d-1}$ and the goal is to establish regularity of \mathcal{J}^α as a function from the unit sphere \mathbb{S}^{d-1} to the space of matrices $M_N(\mathbb{R})$ which, for this purpose, is identified with \mathbb{R}^{N^2} in a usual manner. Let $G^{0,n}(z, \tilde{z})$ be the Green's kernel for the pair $(M^t A^0 M, \mathbb{R}_+^d)$, then by Claim 3.1 and the computations preceding that we have

$$\partial_{\tilde{y}_\alpha} G_{jk}^n(n, \tilde{y}) = \partial_{z_1} G_{jk}^{0,n}(e_d, M^t \tilde{y})m_{\alpha 1} + \dots + \partial_{z_d} G_{jk}^{0,n}(e_d, M^t \tilde{y})m_{\alpha d}.$$

Using this we make a change of variables in (3.5) by the formula $\tilde{y} = M\tilde{z}$, where $\tilde{z} \in \mathbb{R}_+^d$. As $G^{0,n}$ has zero boundary conditions with respect to both variables, we get that all tangential derivatives in the last expression are vanishing. Also, since $Me_d = n$ it follows that $m_{\alpha d} = n_\alpha$ for any $1 \leq \alpha \leq d$. We thus get

$$(3.6) \quad \mathcal{J}^\alpha(n) = n_\alpha \int_{\partial\mathbb{R}_+^d} \partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z}) d\sigma(\tilde{z}).$$

The following bound is proved in [9, estimate (2.17) of Lemma 2.5]

$$|G^{0,n}(z, \tilde{z})| \leq C \frac{z_d \tilde{z}_d}{|z - \tilde{z}|^d}, \quad z \neq \tilde{z} \text{ in } \mathbb{R}_+^d,$$

where C is independent of n . Since $G^{0,n}(e_d, \cdot)$ is zero on $\partial\mathbb{R}_+^d$, from the last estimate it easily follows that $|\nabla_{\tilde{z}} G^{0,n}(e_d, \tilde{z})| \leq C|e_d - \tilde{z}|^{-d}$, for all $\tilde{z} \in \partial\mathbb{R}_+^d$, and hence the integral in (3.6) is absolutely convergent, and is uniformly bounded with respect to n .

Remark 3.2. *Observe, that while $\mathcal{J}^\alpha(n)$ is independent of the orthogonal matrix M , the kernel $G^{0,n}(z, \tilde{z})$ implicitly depends on M . For the objective of this section the choice of M is irrelevant, and for the clarity of notation we do not incorporate it into the notation for $G^{0,n}$. However, in the analysis of regularity of kernels $G^{0,n}$ with respect to n , the choice of M plays a key role. The choice of rotation matrices is discussed in subsection A.1. It is interesting to observe, that whereas the integral of $G^{0,n}$ is easily seen to be smooth with respect to n , proving a similar result for $G^{0,n}$ itself is comparatively more involved, and contains some topological nuances briefly discussed in the Appendix.*

We finish this section with the following result.

Lemma 3.3. *For any $1 \leq \alpha \leq d$ each component of the matrix function $\mathcal{J}^\alpha(n) : \mathbb{S}^{d-1} \rightarrow M_N(\mathbb{R})$ is a smooth real-valued function on \mathbb{S}^{d-1} .*

Proof. Set $\mathcal{J}(n) = \int_{\partial\mathbb{R}_+^d} \partial_{\tilde{z}_d} G^{0,n}(e_d, \tilde{z}) d\sigma(\tilde{z})$, clearly it is enough to prove the claim for the matrix-function $\mathcal{J}(n)$. In view of Claim 3.1 the coefficient tensor corresponding to $G^{0,n}$ is $M^t A^0 M =: B^0$. Next, referring to [18, p. 358], we know that the Poisson's kernel $P^{0,n} = (P_{ij}^{0,n})_{i,j=1}^N$ corresponding to $G^{0,n}$ is defined by

$$P_{ij}^{0,n}(z, \tilde{z}) = -B_{kj}^{0,\alpha\beta} \partial_{\tilde{z}_\alpha} G_{ik}^{0,n}(z, \tilde{z})(e_d)_\beta, \quad z \in \mathbb{R}_+^d, \tilde{z} \in \partial\mathbb{R}_+^d.$$

Since $G^{0,n}$ has zero boundary conditions in \mathbb{R}_+^d with respect to both of its variables, all tangential derivatives in the last expression are vanishing, and as $(e_d)_\beta = \delta_{\beta d}$, for all $1 \leq i, j \leq N$ we obtain

$$(3.7) \quad P_{ij}^{0,n}(z, \tilde{z}) = -B_{kj}^{0,dd} \partial_{\tilde{z}_d} G_{ik}^{0,n}(z, \tilde{z}).$$

From definitions of B^0 and M , for each fixed $1 \leq k, j \leq N$ we have

$$B_{kj}^{0,dd} = e_d^t B_{kj} e_d = e_d^t M^t A_{kj}^0 M e_d = (M e_d)^t A_{kj}^0 (M e_d) = n^t A_{kj}^0 n \in \mathbb{R}.$$

Combining this with (3.7), for the (i, j) -th entry of the matrix $P^{0,n}$ we get

$$(3.8) \quad P_{ij}^{0,n}(z, \tilde{z}) = -n^t A_{kj}^0 n \partial_{\tilde{z}_d} G_{ik}^{0,n}(z, \tilde{z}).$$

For $n \in \mathbb{S}^{d-1}$ consider the matrix $A(n) = (a_{kj}(n))_{k,j=1}^N$, where we have set $a_{kj}(n) = -n^t A_{kj}^0 n$. Now, observe that for column-vector $v_i = (0, \dots, 1, \dots, 0)^t \in \mathbb{R}^N$ with 1 on the i -th position, and 0 otherwise, we have $\int_{\partial\mathbb{R}_+^d} P^{0,n}(e_d, \tilde{z}) v_i d\sigma(\tilde{z}) = v_i$ for all $n \in \mathbb{S}^{d-1}$, and any $1 \leq i \leq N$. This follows from that fact that the unique smooth solution to Dirichlet problem has Poisson integral representation. From here and (3.8) we get

$$I_N = \int_{\partial\mathbb{R}_+^d} P^{0,n}(e_d, \tilde{z}) d\sigma(\tilde{z}) = \mathcal{J}(n) A(n),$$

where as before I_N is the $N \times N$ identity matrix. It follows that the matrix $A(n)$ is invertible for any $n \in \mathbb{S}^{d-1}$, and hence $\mathcal{J}(n) = (A(n))^{-1}$. On the other hand all components of $A(n)$ are obviously smooth functions on \mathbb{S}^{d-1} , therefore the determinant of $A(n)$ stays away from 0 by compactness of \mathbb{S}^{d-1} . We conclude that each component of the inverse $(A(n))^{-1}$ is C^∞ on \mathbb{S}^{d-1} , hence we get the claim for \mathcal{J} and finish the proof of the lemma. \square

4. THE REGULARITY OF g^*

The aim of this section is to prove Theorem 1.2. Observe, that so far we had no recourse to assumption (A5) regarding the layered structure, and it is here that it will play a central role in the analysis.

4.1. Change of variables. At several places in this section we will switch from one variable to another; we record the necessary details here. Let $y \in \mathbb{R}^d$ and for a coefficient tensor $B = B^{\alpha\beta}(y) \in M_N(\mathbb{R})$ which is smooth and elliptic in a sense of standard assumptions (A2) and (A3) of Section 1 consider the operator $\mathcal{L} = -\nabla_y \cdot B(y) \nabla_y$. For $x \in \mathbb{R}^d$ set $y = Tx$, where $T \in M_d(\mathbb{R})$ and has non-zero determinant. One may easily deduce that

$$(4.1) \quad \nabla_y = (T^t)^{-1} \nabla_x = (T^{-1})^t \nabla_x.$$

For $1 \leq i, j \leq N$ let B_{ij} be the $d \times d$ matrix formed from the (i, j) -th entries of the matrices $B^{\alpha\beta}$. Then using (4.1) we see that the operator \mathcal{L} in the new variable x can be written as $\mathcal{L} = -\nabla_x \cdot \tilde{B}(Tx) \nabla_x$, where correspondingly

$$(4.2) \quad \tilde{B}_{ij}(Tx) = T^{-1} B_{ij}(Tx) (T^{-1})^t$$

for all $1 \leq i, j \leq N$. To keep track of the ellipticity constant of the new operator we take a family of vectors $\xi = \xi^\alpha \in \mathbb{R}^N$, set $\omega_i = (\xi_i^1, \dots, \xi_i^d)^t$ where $1 \leq i \leq N$ and compute

$$(4.3) \quad \tilde{B}_{ij}^{\alpha\beta} \xi_j^\beta \xi_i^\alpha = \omega_i^t \tilde{B}_{ij} \omega_j = \omega_i^t T^{-1} B_{ij} (T^{-1})^t \omega_j = [(T^{-1})^t \omega_i]^t B_{ij} (T^{-1})^t \omega_j \geq \lambda_B \sum_{i=1}^N \|(T^{-1})^t \omega_i\|^2 \geq \lambda_B \sigma_{\min}^2(T^{-1}) \omega_i \cdot \omega_i = \lambda_B \sigma_{\min}^2(T^{-1}) \xi^\alpha \cdot \xi^\alpha,$$

where λ_B is the ellipticity constant of the original operator and $\sigma_{\min}(T^{-1})$ is the least singular value of the matrix T^{-1} , that is the square root of the smallest eigenvalue of $T^{-1}(T^{-1})^t$. In particular, it follows that the new operator is elliptic, with possibly a different ellipticity constant.

4.2. Solutions with exponentially decaying gradients. Here we use assumption (A5) to gain some extra control on solutions to boundary layer systems. To illustrate what one can get from (A5) we will start with a simple example involving the Laplace operator.

Example 4.1. Assume $N = 1$, i.e we have only one equation, and for an *irrational* direction $n \in \mathbb{S}^{d-1}$ and $u_0 \in C^\infty(\mathbb{T}^d)$ consider the following problem

$$(4.4) \quad \Delta u = 0 \text{ in } \Omega_n \quad \text{and} \quad u = u_0 \text{ on } \partial\Omega_n.$$

Let $\{c_\xi(u_0)\}_{\xi \in \mathbb{Z}^d}$ be the sequence of Fourier coefficients of u_0 . Then, by a direct computation one can easily check that the function

$$(4.5) \quad u(y) = \sum_{\xi \in \mathbb{Z}^d} c_\xi(u_0) e^{-2\pi [|\xi|^2 - (n \cdot \xi)^2]^{\frac{1}{2}} (y \cdot n)} e^{2\pi i \xi \cdot [y - n(y \cdot n)]}, \quad y \in \overline{\Omega}_n,$$

solves (4.4) and satisfies all requirements of Theorem 2.1, where as before $\Omega_n = \{x \in \mathbb{R}^d : x \cdot n > 0\}$. It follows in particular that u defined by (4.5) is the unique solution of (4.4) given by Theorem 2.1. Since $n \notin \mathbb{R}\mathbb{Q}^d$, the equality case of the Cauchy-Schwarz inequality provides $|\xi|^2 - (n \cdot \xi)^2 \neq 0$ unless $\xi = 0$ and hence the boundary layer tail in this case is simply $c_0(u_0)$.

Now assume that u_0 is independent of the last coordinate, i.e. $(e_d \cdot \nabla) u_0 = 0$ on \mathbb{T}^d . This condition can be reformulated in terms of Fourier coefficients. Namely, using the smoothness of u_0 and applying $e_d \cdot \nabla$ on the Fourier series of u_0 , by Parseval's identity we obtain that $\xi_d c_\xi(u_0) = 0$ for all $\xi \in \mathbb{Z}^d$. The latter implies that $c_\xi(u_0) = 0$ for any $\xi \in \mathbb{Z}^d$

with $\xi_d \neq 0$, that is the Fourier spectrum of u_0 is contained in the sublattice $\mathbb{Z}^{d-1} \times \{0\} \subset \mathbb{Z}^d$. Next, suppose the vector n satisfies $n_d \neq 0$. Then for $\xi = (\xi', 0) \in \mathbb{Z}^{d-1} \times \{0\}$ we have

$$(n \cdot \xi)^2 \leq (n_1^2 + \dots + n_{d-1}^2) |\xi'|^2 = (1 - n_d^2) |\xi|^2,$$

therefore

$$||\xi|^2 - (n \cdot \xi)^2| = |\xi|^2 \left| 1 - \frac{(n \cdot \xi)^2}{|\xi|^2} \right| \geq n_d^2 |\xi|^2.$$

The latter combined with (4.5) illustrates that given the special structure of the Fourier spectrum of u_0 , the solution of (4.4) converges exponentially fast in the direction of the normal vector n toward its boundary layer tail. Also, it is clear that the decay properties deteriorate as $n_d \rightarrow 0$. It should also be noted that while u_0 was independent of e_d , the solution u does not necessarily satisfy this independence criterion.

To treat the general case we will need a construction due to L. Tartar. For Y' , an open parallelepiped in \mathbb{R}^{d-1} , set $G = Y' \times (0, \infty)$. Let $f = \{f_i\}$ and $F = \{F_i^\alpha\}$ be given smooth functions, where $1 \leq i \leq N$ and $1 \leq \alpha \leq d$. For the unknown vector $u = (u_1, \dots, u_N)$ consider the following problem

$$(4.6) \quad \begin{cases} -\nabla \cdot A(y) \nabla u(y) = f - \nabla \cdot F(y), & \text{in } G, \\ u(y', 0) = 0, & y' \in Y', \\ u(\cdot, y_d), & \text{is } Y'\text{-periodic for any } y_d > 0, \end{cases}$$

where the system of equations is understood as follows

$$-\frac{\partial}{\partial y_\alpha} \left(A_{ij}^{\alpha\beta}(y) \frac{\partial u_j}{\partial y_\beta}(y) \right) = f_i - \frac{\partial F_i^\alpha}{\partial y_\alpha}, \quad i = 1, 2, \dots, N.$$

We assume that there exists $\tau_0 > 0$ such that

$$(4.7) \quad e^{\tau_0 y_d} f(y) \in L^2(G; \mathbb{R}^N) \quad \text{and} \quad e^{\tau_0 y_d} F(y) \in L^2(G; \mathbb{R}^{d \times N}),$$

and

$$(4.8) \quad f(\cdot, y_d) \text{ and } F(\cdot, y_d) \text{ are both } Y'\text{-periodic for any } y_d \geq 0.$$

In order to clarify the periodicity condition in (4.6), recall the definition of $H_{per}^1(Y')$, which is the closure with respect to H^1 -norm of the space of smooth and Y' -periodic functions. In particular, functions in $H_{per}^1(Y')$ have equal traces on opposite faces of Y' . Now for $\tau > 0$ set

$$V_\tau(G) = \{v \in H_{loc}^1(G) : v \in L_{loc}^2(\mathbb{R}_+; H_{per}^1(Y')), e^{\tau y_d} \nabla v(y) \in L^2(G) \text{ and } v(y', 0) \equiv 0\}.$$

One can see that V_τ is a Hilbert space with scalar product defined by

$$[u, v]_\tau = \int_G e^{2\tau y_d} \nabla u(y) \cdot \nabla v(y) dy.$$

The norm on $V_\tau(G)$ induced from the scalar product is denoted by $\|\cdot\|_{V_\tau(G)}$. The existence of solutions to (4.6) with exponentially decaying gradients is given in the following result.

Theorem 4.2. (see [19, Chapter 18], and [16, Theorem 10.1]) *Assume (4.7), (4.8) and that the coefficient tensor in (4.6) is bounded and is uniformly elliptic with ellipticity constant $\lambda_A > 0$. Then for any $0 < \tau < \min\{\tau_0, \frac{\lambda_A}{2\|A\|_\infty}\}$ there exists a unique solution u to system (4.6) in the space $V_\tau(G)$. Moreover, for any such τ one has the estimate*

$$(4.9) \quad \|u\|_{V_\tau(G)} \leq \frac{C}{\tau} \frac{1}{\lambda_A - 2\tau\|A\|_\infty} \left[\|e^{\tau y_d} f\|_{L^2(G; \mathbb{R}^N)} + \|e^{\tau y_d} F\|_{L^2(G; \mathbb{R}^{d \times N})} \right],$$

where the constant C depends on dimension d and the parallelepiped Y' .

Observe, that at this stage we do not use periodicity of A , nor any other structural restriction is imposed on the operator.

Remark 4.3. *The formulation of Theorem 4.2 is slightly more general than the original one as given e.g. in [16] or [19]. Namely, here it is stated for elliptic systems rather than scalar equations, and involves detailed estimates of V_τ norms of solutions. The proof however, follows the lines of the original proof with small changes to deal with systems of equations, and making the norm estimate of u explicit.*

The following useful fact follows directly from Theorem 4.2.

Corollary 4.4. *Assume the coefficient tensor $A(y)$ is bounded, uniformly elliptic with ellipticity constant $\lambda_A > 0$, smooth and Y' -periodic for each fixed $y_d \geq 0$. Then, for any smooth and Y' -periodic vector-function g with values in \mathbb{R}^N the following problem*

$$(4.10) \quad \begin{cases} -\nabla \cdot A(y) \nabla u(y) = 0, & y \in \mathbb{R}_+^d, \\ u(y', 0) = g(y'), & y' \in \mathbb{R}^{d-1} \end{cases}$$

has a unique weak solution $u \in H_{loc}^1(\mathbb{R}_+^d)$ with the properties

$$(4.11) \quad u \in L_{loc}^2(\mathbb{R}_+; H_{per}^1(Y')) \quad \text{and} \quad e^{\tau y_d} \nabla u \in L^2(Y' \times \mathbb{R}_+^d) \quad \text{for any} \quad 0 < \tau < \frac{\lambda_A}{2\|A\|_\infty}.$$

Moreover, the solution u satisfies

$$(4.12) \quad \|e^{\tau y_d} \nabla u\|_{L^2(G; \mathbb{R}^{d \times N})} \leq \frac{C}{\tau} \frac{\|A\|_\infty}{\lambda_A - 2\tau\|A\|_\infty} \|g\|_{H^1(Y'; \mathbb{R}^N)}.$$

Proof. Fix any non-negative and compactly supported smooth function $\varphi : \mathbb{R} \rightarrow [0, 1]$ such that $\varphi = 1$ near 0. Using the cut-off φ we lift the boundary data g into \mathbb{R}_+^d by setting $\tilde{g}(y) = \varphi(y_d)g(y')$ for all $y = (y', y_d) \in G$. Since \tilde{g} has compact support in the direction of e_d we have that $e^{\tau y_d} \nabla \tilde{g} \in L^2(G; \mathbb{R}^{d \times N})$, for any $\tau > 0$, and we let \tilde{u} be the unique solution to (4.6) with the right-hand side $\nabla \cdot A(y) \nabla \tilde{g}$ given by Theorem 4.2. Next, we denote by \tilde{u}_{per} the extension of \tilde{u} to \mathbb{R}_+^d by periodicity in tangential variables. More precisely for any $(y', y_d) \in \mathbb{R}_+^d$ we set $\tilde{u}_{per}(y', y_d) = \tilde{u}(y' - \xi', y_d)$ where ξ' is the unique element of \mathbb{Z}^{d-1} with the property that $y' - \xi' \in Y'$. It then follows by standard arguments that we have $\tilde{u}_{per} \in H_{loc}^1(\mathbb{R}_+^d)$ for the extension and that \tilde{u}_{per} defines a weak solution to⁵

$$(4.13) \quad -\nabla \cdot A \nabla \tilde{u}_{per} = \nabla \cdot A \nabla \tilde{g} \quad \text{in} \quad \mathbb{R}_+^d \quad \text{and} \quad \tilde{u}_{per} = 0 \quad \text{on} \quad \partial \mathbb{R}_+^d.$$

Since \tilde{u}_{per} solves (4.13) the function $u = \tilde{u}_{per} + \tilde{g}$ satisfies all requirements of the corollary, and the estimate (4.12) follows easily from the corresponding estimate of Theorem 4.2. \square

⁵For reader's convenience we briefly sketch the argument. First, the inclusion $\tilde{u}_{per} \in H_{loc}^1(\mathbb{R}_+^d)$ is a direct corollary to the fact that $\tilde{u}(\cdot, y_d) \in H_{per}^1(Y')$ for any $y_d \geq 0$ (see e.g. [6, Proposition 3.50] for a similar treatment). Next, to see that \tilde{u}_{per} solves (4.13) it is enough to see that \tilde{u}_{per} defines a solution across lateral boundary of G , i.e. $\omega := \partial Y' \times \mathbb{R}_+$. Writing the definition of weak solution to (4.13) (i.e. testing the equation against C_0^∞ functions) we see that it suffices to have $\tilde{u} \in H^2$ locally in a neighbourhood of each point of ω (so that to make sense of the trace of the derivatives of \tilde{u}_{per}) and $\nabla \tilde{u}(\cdot, y_d) \in H_{per}^1(Y')$ for any $y_d \geq 0$, as then equality in (4.13) will simply follow by localizing the equation in a neighbourhood of ω and doing partial integration in the weak (integral) formulation. For the H^2 -regularity, we first see that tangential derivatives of \tilde{u} solve a similar problem in $V_\tau(G)$ as \tilde{u} itself (by considering difference quotients instead to be more precise), which shows that tangential derivatives of u have the desired regularity and periodicity properties. After having treated the tangential derivatives, the d -th derivative of \tilde{u} can be handled from the system itself, by separating the term with dd -th derivative, and treating the rest as lower-order terms. Namely, one can write $A_{ij}^{dd} \partial_{dd}^2 \tilde{u}_j \in L_{loc}^2(\mathbb{R}_+; H_{per}^1(Y'))$, where $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_N)$, and then invert the $N \times N$ matrix on the left-hand side (relying on ellipticity of A) to get the mentioned regularity and periodicity properties of $\partial_{dd} \tilde{u}$ (cf. [10, eq. (2.12)], where the situation is more complicated due to the lack of uniform ellipticity).

4.3. The case when $\nu_0 = e_d$. We will first carry out the analysis when the vector ν_0 defined from assumption (A5) coincides with $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. To fix the ideas, we let $A = A^{\alpha\beta} \in M_N(\mathbb{R})$ be a coefficient tensor satisfying the standard ellipticity, smoothness, and periodicity conditions of Section 1, and in addition we require A to be independent of e_d , or equivalently the d -th coordinate. We also fix $v_0 \in C^\infty(\mathbb{T}^d)$ which is assumed to be independent of e_d as well. Then, for a given $n \in \mathbb{S}^{d-1}$ consider the following problem

$$(4.14) \quad \begin{cases} -\nabla_y \cdot A(y) \nabla_y v(y) = 0, & y \in \Omega_n, \\ v(y) = v_0(y), & y \in \partial\Omega_n. \end{cases}$$

Let v be the unique solution to (4.14) given by Theorem 2.1. The aim now is to show that this solution has some extra regularity properties given the structural restriction on A and v_0 .

We will assume that $n_d \neq 0$, and then without loss of generality will take $n_d > 0$, as the case $n_d < 0$ works in the same way. The case of $n_d = 0$ is degenerate, and the analysis breaks down. Also, notice that at this stage we do not require n to be irrational. To the unit vector $n = (n_1, \dots, n_d)$ we attach a matrix $T_n \in M_d(\mathbb{R})$ given by

$$(4.15) \quad T_n = \begin{pmatrix} & & & 0 \\ & I_{d-1} & & \vdots \\ & & & 0 \\ -\frac{n_1}{n_d} & \dots & -\frac{n_{d-1}}{n_d} & 1 \end{pmatrix},$$

where $I_{d-1} \in M_{d-1}(\mathbb{R})$ is the identity. It is clear that a linear transformation associated with T_n is a bijection from $\overline{\mathbb{R}_+^d}$ to $\overline{\Omega_n}$, and that

$$(4.16) \quad T_n^{-1} = \begin{pmatrix} & & & 0 \\ & I_{d-1} & & \vdots \\ & & & 0 \\ \frac{n_1}{n_d} & \dots & \frac{n_{d-1}}{n_d} & 1 \end{pmatrix}$$

is the inverse of T_n . We make a change of variables in (4.14) by setting $y = T_n z$, where $z \in \mathbb{R}_+^d$. Following the notation and results of Section 4.1, if we let A_n be the coefficient tensor in the new variable z then

$$(4.17) \quad (A_n)_{ij}(T_n z) = T_n^{-1} A_{ij}(T_n z) (T_n^{-1})^t = T_n^{-1} A_{ij}(z) (T_n^{-1})^t.$$

The last equality of (4.17) follows from the fact that the linear transformation T_n acts as an identity on the first $d-1$ variables, and affects only the d -th coordinate on which A has no dependence by assumption. A similar reasoning applied to v_0 gives $v_0(T_n z) = v_0(z)$. In particular, the change of variable by T_n leaves periodicity of the operator and the boundary data invariant. Thus, the problem (4.14) is being transformed to

$$(4.18) \quad \begin{cases} -\nabla_z \cdot A_n(z) \nabla_z w(z) = 0, & z \in \mathbb{R}_+^d, \\ w(z) = v_0(z), & z \in \partial\mathbb{R}_+^d, \end{cases}$$

where A_n is given by (4.17) and $w(z) = v(T_n z)$. The ellipticity of A_n follows from non-degeneracy of T_n and (4.3). We now give an estimate on the ellipticity constant of A_n which we will use in the sequel. Following (4.3) we need to bound the smallest singular value of T_n^{-1} from below, which is being done using the following result.

Theorem 4.5. (see [12, Theorem 1]) *For a matrix $T \in M_d(\mathbb{C})$ let $r_i(T)$ be the Euclidean norm of its i -th row, $c_i(T)$ be the Euclidean norm of its i -th column, and set $r_{\min}(T) =$*

$\min_{1 \leq i \leq d} r_i(T)$ and $c_{\min}(A) = \min_{1 \leq i \leq d} c_i(T)$. Then, for $\sigma_{\min}(T)$, the smallest singular value of T , one has

$$(4.19) \quad \sigma_{\min}(T) \geq \left(\frac{d-1}{d}\right)^{(d-1)/2} |\det T| \max \left\{ \frac{c_{\min}(T)}{\prod_{i=1}^d c_i(T)}, \frac{r_{\min}(T)}{\prod_{i=1}^d r_i(T)} \right\}.$$

We have $\det(T_n^{-1}) = 1$ and using the fact that $|n| = 1$ we obtain $r_{\min}(T_n^{-1})/\prod_{i=1}^d r_i(T_n^{-1}) = n_d$. Now by virtue of (4.19) it follows that

$$(4.20) \quad \sigma_{\min}(T_n^{-1}) \geq \left(\frac{d-1}{d}\right)^{(d-1)/2} n_d.$$

Hence, for λ_{A_n} , the ellipticity constant of the operator in (4.18), we have by (4.3) and (4.20) that

$$(4.21) \quad \lambda_{A_n} \geq c_d \lambda_A n_d^2,$$

where c_d is a constant depending on the dimension, and λ_A is the ellipticity constant of the original operator. Invoking Corollary 4.4 we let w_n be the unique weak solution of (4.18) with finite $\|\cdot\|_{V_\tau(G)}$ -norm for all $0 < \tau < \frac{\lambda_{A_n}}{2\|A_n\|_\infty}$ where $G = (0, 1)^{d-1} \times \mathbb{R}_+$. Since the coefficients and the boundary data are smooth in (4.18) it follows from the standard elliptic regularity that $w_n \in C^\infty(\overline{\mathbb{R}_+^d})$ (see e.g. [11, Corollary 4.12] and [11, Theorem 5.21]). Moreover, we have by construction that $w_n(\cdot, z_d)$ is \mathbb{Z}^{d-1} -periodic for each $z_d \geq 0$ and has exponentially decaying gradient in the direction of e_d . As w_n solves (4.18) it follows that $v_n(y) = w_n(T_n^{-1}y)$ solves (4.14), where $y \in \overline{\Omega_n}$. We now need to check that v_n coincides with v which was the solution to (4.14) given by Theorem 2.1. For that we will use the next lemma, which, as well as the initial idea of exploiting layered structure of the problem were motivated by [17].

Lemma 4.6. *For $n \in \mathbb{S}^{d-1}$ satisfying $n_d > 0$, let v_n be the solution to (4.14) constructed as above. Then $v_n \in C^\infty(\overline{\Omega_n}) \cap L^\infty(\overline{\Omega_n})$ and satisfies the following properties*

- (a) $\|\nabla v_n\|_{L^\infty(\{y \cdot n > t\})} \rightarrow 0$, as $t \rightarrow \infty$,
- (b) $\int_0^\infty \|(n \cdot \nabla)v_n\|_{L^\infty(\{y \cdot n = t\})}^2 dt < \infty$.

Proof. We have $v_n(y) = w_n(T_n^{-1}y)$ where $y \in \overline{\Omega_n}$, hence the up to the boundary smoothness of v_n directly follows from that of w_n .

Let $G = (0, 1)^{d-1} \times \mathbb{R}_+$ and fix some $\tau > 0$ so that w_n has finite $V_\tau(G)$ -norm. Since w_n solves (4.18), where the coefficients and the boundary data have bounded C^k -norms for any $k \geq 0$, by standard Schauder estimates near the boundary (see [11, Theorem 5.21]) we have that $|\nabla_z w_n(z)| \leq C$ uniformly for all $z = (z', z_d) \in \mathbb{R}^{d-1} \times [0, \infty)$ satisfying $z_d \leq 1$. We now estimate $|\nabla_z w_n(z)|$ for $z = (z', z_d) \in \mathbb{R}_+^d$ with $z_d > 1$. By \mathbb{Z}^{d-1} -periodicity of $w_n(\cdot, z_d)$ we may assume that $z' \in (0, 1)^{d-1}$. For $r > 0$ let $K(z, r)$ be a closed cube centred at z and having side length r . In view of interior Schauder estimates (see [11, Theorem 5.19]) we have

$$(4.22) \quad \|\nabla w_n\|_{L^\infty(K(z, 1/4))} \leq C \|\nabla w_n\|_{L^2(K(z, 1/2))},$$

with constant C independent of z . Set $K'(z, 1/2)$ to be the $(d-1)$ -dimensional cube which is the projection of $K(z, 1/2)$ onto $\mathbb{R}^{d-1} \times \{0\}$. We have

$$(4.23) \quad \begin{aligned} \|\nabla w_n\|_{L^2(K(z, 1/2))}^2 &= \int_{K'(z, 1/2)} \int_{z_d-1/2}^{z_d+1/2} |\nabla w_n(x', x_d)|^2 dx' dx_d \leq \\ &e^\tau e^{-2\tau z_d} \int_{K'(z, 1/2)} \int_{z_d-1/2}^{z_d+1/2} |e^{\tau x_d} \nabla w_n(x', x_d)|^2 dx' dx_d \leq e^\tau e^{-2\tau z_d} \|w_n\|_{V_\tau(G)}^2, \end{aligned}$$

where we have used the periodicity of w to get a bound in $V_\tau(G)$ -norm. From (4.23) and (4.22) we obtain

$$(4.24) \quad |\nabla w_n(z)| \leq C e^{-\tau z_d} \|w_n\|_{V_\tau(G)}, \quad z \in \mathbb{R}_+^d,$$

where we have also included the case of $z_d \leq 1$ in view of the uniform bound on the gradient. Both assertions of the lemma follow directly from (4.24) and the relation $\nabla_y v(y) = (T_n^{-1})^t \nabla_z w_n(z)$, with $y = T_n z$ which is due to the change of variables formula.

Finally, by writing

$$w_n(z) = w_n(z', z_d) = v_0(z', 0) + \int_0^{z_d} \partial_t w_n(z', t) dt$$

and using (4.24) we get $w \in L^\infty(\overline{\Omega_n})$, and complete the proof of the lemma. \square

By Lemma 4.6, v_n gives a smooth and bounded solution to (4.14), and satisfies condition 1 of Theorem 2.1. But the solution with these properties is unique according to Theorem 2.1. Hence we have the following.

Corollary 4.7. *Fix $n \in \mathbb{S}^{d-1}$ such that $n_d > 0$, and assume that in (4.14) A and v_0 are independent of e_d and satisfy the usual ellipticity, smoothness, and periodicity assumptions of Section 1. Then the solution v_n of (4.14) coincides with the one given by Theorem 2.1.*

From the properties of w_n we now deduce an expansion for v_n . For $n \in \mathbb{S}^{d-1}$ satisfying $n_d > 0$ let w_n be the solution to (4.18) constructed as above. Then due to the periodicity condition we have

$$(4.25) \quad w_n(z) = w_n(z', z_d) = \sum_{\xi' \in \mathbb{Z}^{d-1}} c_{\xi'}(n; z_d) e^{2\pi i \xi' \cdot z'} = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi(n; z_d) e^{2\pi i \xi \cdot z},$$

where $(z', z_d) \in \mathbb{R}^{d-1} \times [0, \infty)$ and for $\xi = (\xi', 0) \in \mathbb{Z}^{d-1} \times \{0\}$ we let

$$(4.26) \quad c_\xi(n; z_d) = \int_{\mathbb{T}^{d-1}} w_n(z', z_d) e^{-2\pi i \xi \cdot z} dz'$$

be the ξ -th Fourier coefficient of $w_n(\cdot, z_d)$. By the construction of w_n for any $\xi \in \mathbb{Z}^{d-1} \times \{0\}$ we have

$$(4.27) \quad c_\xi(n; 0) = c_\xi(v_0), \quad \forall n \in \mathbb{S}^{d-1} \text{ satisfying } n_d > 0,$$

where $c_\xi(v_0)$ is the corresponding Fourier coefficient of the fixed boundary data v_0 involved in (4.14). The definition of T_n yields

$$z_d = T_n^{-1} y \cdot e_d = y \cdot (T_n^{-1})^t e_d = \frac{y \cdot n}{n_d},$$

and $z' = y'$. Using these relations between y and z , from (4.25) for the solution of v_n of (4.14) we obtain

$$(4.28) \quad v_n(y) = w_n(T_n^{-1}y) = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi \left(n; \frac{y \cdot n}{n_d} \right) e^{2\pi i \xi \cdot y}, \quad y \in \overline{\Omega_n}.$$

Observe that in view of the smoothness of w_n the function $t \mapsto c_\xi(n; t) \in \mathbb{R}^N$ is smooth on $[0, \infty)$ for each $\xi \in \mathbb{Z}^{d-1} \times \{0\}$. What we show next is a stability result with respect to normal vector n for the derivative of this function.

Lemma 4.8. *Fix $\delta > 0$ small, and let $\nu, \mu \in \mathbb{S}^{d-1}$ satisfy $\nu_d, \mu_d \geq \delta$. Then there exists a constant $C_\delta = C(\delta, A)$ such that for any $t \geq 0$ and all $\xi \in \mathbb{Z}^{d-1} \times \{0\}$ one has*

$$(a) \quad \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} |\partial_t c_\xi(\nu; t)| < \infty$$

$$(b) \quad |\partial_t c_\xi(\nu; t) - \partial_t c_\xi(\mu; t)| \leq C_\delta |\nu - \mu| \times \|v_0\|_{C^2(\mathbb{T}^d)}$$

where $c_\xi(n; t)$ is given by (4.26).

Proof. We start with part (a). From (4.25) we have that $\partial_t c_\xi(\nu; t)$ is the ξ -th Fourier coefficient of $\partial_d w_n(\cdot, t)$, which is a smooth and \mathbb{Z}^{d-1} -periodic function by construction. Hence, we have (a).

In order to establish stability estimate (b), observe that thanks to (4.26) it suffices to prove stability of ∇w_n with respect to n . Recall the notation $G = \mathbb{T}^{d-1} \times \mathbb{R}_+$, and set $u = w_\nu - w_\mu$. We get that u is a smooth solution to

$$(4.29) \quad \begin{cases} -\nabla \cdot A_\nu(z) \nabla u(z) = \nabla \cdot F(z), & z \in \mathbb{R}_+^d, \\ u(z', 0) = 0, & z' \in \mathbb{R}^{d-1}, \end{cases}$$

where we have denoted $F(z) := (A_\nu(z) - A_\mu(z)) \nabla w_\mu(z)$. Observe that $u(\cdot, z_d)$, as well as $F(\cdot, z_d)$ are periodic with respect to \mathbb{Z}^{d-1} for any $z_d \geq 0$. Also, due to the construction it follows that $u \in V_\tau(G)$ for some $\tau > 0$ which will be specified in a moment. Since solution to (4.29) is unique in the space V_τ , we may apply estimate (4.9) of Theorem 4.2 and by so obtain

$$(4.30) \quad \|u\|_{V_\tau(G; \mathbb{R}^{d \times N})} \leq \frac{C \|e^{\tau z_d} F(z)\|_{L^2(G; \mathbb{R}^{d \times N})}}{\tau \lambda_{A_\nu} - 2\tau \|A_\nu\|_{L^\infty(\mathbb{R}^d)}},$$

where in (4.30) we are following notation of Theorem 4.2. Using (4.2) and (4.16) from the definition of A_ν we have $\|A_\nu\|_{L^\infty(\mathbb{R}^d)} \leq C \|A\|_{L^\infty(\mathbb{T}^d)} \delta^{-2}$. The latter combined with (4.21) implies that $\tau = \frac{\lambda_A}{4\|A\|_\infty} \delta^4$ is a valid choice in (4.30), where λ_A is the ellipticity constant of the original operator in (4.14). Thus we will keep in mind that we have a uniform control over τ in terms of the threshold δ . Next, by (4.17) and (4.16) we easily get

$$(4.31) \quad \|A_\nu(z) - A_\mu(z)\|_{L^\infty(\mathbb{R}^d)} \leq C_\delta |\nu - \mu|,$$

which in combination with the choice of τ and (4.30) infers

$$(4.32) \quad \|u\|_{V_\tau(G)} \leq C_\delta |\nu - \mu| \times \|w_\mu\|_{V_\tau(G)} \leq C_\delta |\nu - \mu| \times \|v_0\|_{C^1(\mathbb{T}^d)},$$

where the second inequality in (4.32) is due to (4.12).

Now fix some $z_0 \in \partial \mathbb{R}_+^d$, and for $r > 0$ denote by $\mathcal{K}(z_0, r)$ the intersection of a cube with side length r and center at z_0 with \mathbb{R}_+^d . By boundary Schauder estimates (see [11, Theorem 5.21] and its proof) we have

$$(4.33) \quad \|\nabla u\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1/2))} \lesssim_\delta \|\nabla u\|_{L^2(\mathcal{K}(z_0, 1))} + \|F\|_{C^{0,\sigma}(\mathcal{K}(z_0, 1))},$$

where $0 < \sigma < 1$ is any fixed parameter, and the dependence of the constant in the inequality on parameter δ comes from the dependence of the ellipticity constant of A_ν on δ . It is clear that

$$(4.34) \quad \|\nabla u\|_{L^2(\mathcal{K}(z_0,1))} \leq \|u\|_{V_\tau(G)}.$$

Next, using the definition of F we have

$$(4.35) \quad \|F\|_{C^{0,\sigma}(\mathcal{K}(z_0,1))} \leq \|A_\nu(z) - A_\mu(z)\|_{C^{0,\sigma}(\mathcal{K}(z_0,1))} \|\nabla w_\mu\|_{C^{0,\sigma}(\mathcal{K}(z_0,1))}.$$

The first factor in the right-hand side of (4.35) is easily seen, as in (4.31), to be bounded by $C_\delta|\nu - \mu|$. For the second one, we do a recourse to the construction of w_μ in Corollary 4.4 and again using Schauder estimates at the boundary we get

$$\|\nabla w_\mu\|_{C^{0,\sigma}(\mathcal{K}(z_0,1))} \lesssim_\delta \|\nabla w_\mu\|_{L^2(\mathcal{K}(z_0,2))} + \|A_\mu \nabla v_0\|_{C^{0,\sigma}(\mathcal{K}(z_0,2))} + \|v_0\|_{C^2(\mathcal{K}(z_0,2))}.$$

In the last expression we estimate the L^2 -norm of the gradient of w_μ by V_τ norm, which, on its turn, is controlled by (4.12). Getting back to (4.35) we obtain

$$(4.36) \quad \|F\|_{C^{0,\sigma}(\mathcal{K}(z_0,1))} \leq C_\delta \|v_0\|_{C^2(\mathbb{T}^d)} |\nu - \mu|.$$

We now use (4.36), (4.34) and (4.32) in (4.33) to get

$$(4.37) \quad \|\nabla u\|_{L^\infty(\mathcal{K}(z_0,1/2))} \leq C_\delta \|v_0\|_{C^2(\mathbb{T}^d)} |\nu - \mu|.$$

The claim (b) of the lemma now follows directly by taking the derivative under the integral sign in (4.26) and applying (4.37). The proof is complete. \square

4.4. Boundary layer correctors. For irrational direction $n \in \mathbb{S}^{d-1}$ satisfying $n \cdot \nu_0 > 0$, and for fixed $1 \leq \gamma \leq d$ let $v_n^{*,\gamma}$ be the solution to (2.3) in a sense of Theorem 2.1. Under assumption (A5) on the operator we apply $\nu_0 \cdot \nabla$ on both sides of the system in (2.4) and get that $\chi^{*,\gamma}$, the solution to the cell-problem, is also independent of ν_0 . We next fix a $d \times d$ matrix T_0 with integer entries such that $T_0 e_d = \nu_0$ and⁶ $\det T_0 \neq 0$. Making a change of variables in (2.3) by setting $y = T_0 z$, and observing that $y \cdot n = z \cdot T_0^t n$, we transform the problem for boundary layer corrector to

$$(4.38) \quad \begin{cases} -\nabla_z \cdot \tilde{A}(T_0 z) \nabla_z \tilde{v}_n^\gamma(z) = 0, & z \in \Omega_{T_0^t n}, \\ \tilde{v}_n^\gamma(z) = -\tilde{\chi}^\gamma(z), & z \in \partial\Omega_{T_0^t n}, \end{cases}$$

where we have set $v_n^{*,\gamma}(T_0 z) = \tilde{v}_n^\gamma(z)$, $\chi^{*,\gamma}(T_0 z) = \tilde{\chi}^\gamma(z)$, and the coefficients are being transformed as in Section 4.1. By the formula (4.1) we have

$$\nu_0 \cdot \nabla_y = T_0 e_d \cdot (T_0^{-1})^t \nabla_z = e_d \cdot \nabla_z,$$

hence both the operator and the boundary data in (4.38) are independent of the d -th coordinate. Moreover, as T_0 has integer entries, it follows that coefficients of (4.38) as well as the boundary data are periodic with respect to \mathbb{Z}^d . It is also clear that by the irrationality of n and the choice of T_0 we have $T_0^t n \notin \mathbb{R}\mathbb{Q}^d$. Also, $\tilde{v}_n^\gamma(z)$ is the solution of (4.38) in a sense of Theorem 2.1 if and only if $v_n^{*,\gamma}(T_0 z)$ is the solution to (4.38) in a sense of Theorem 2.1. Finally noticing that $T_0^t n \cdot e_d = n \cdot \nu_0 > 0$, in (4.38) we are now in

⁶For our arguments it is enough to have existence of the inverse of T_0 with rational entries, however, it is useful to see that with a little extra work one may assure $\det T_0 = 1$ provided the greatest common divisor of the components of ν_0 equals one (see Claim A.2). The latter can always be assumed without loss of generality, as the condition (A5) is invariant under scaling of ν_0 . The advantage of having $\det T_0 = 1$ lies in the fact that the inverse of T_0 will also have integer entries, which ensures that all boundary layer correctors $v_n^{*,\gamma}$ defined in (4.40) remain periodic with respect to \mathbb{Z}^{d-1} in tangential directions.

a position to apply the analysis of Section 4.3. In particular, from (4.28) we get that \tilde{v}_n^γ , the solution to (4.38), has the following expansion

$$\tilde{v}_n^\gamma(z) = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi^\gamma \left(T_0^t n; \frac{z \cdot T_0^t n}{T_0^t n \cdot e_d} \right) e^{2\pi i \xi \cdot z}, \quad z \in \overline{\Omega_{T_0^t n}},$$

where Fourier coefficients c_ξ^γ are defined in analogy with (4.26), in particular we have

$$(4.39) \quad c_\xi^\gamma(T_0^t n; 0) = c_\xi(-\chi^{*,\gamma}), \quad \xi \in \mathbb{Z}^{d-1} \times \{0\}.$$

Since $T_0^{-1} y \cdot T_0^t n = y \cdot n$ and $T_0^t \cdot e_d = n \cdot \nu_0$ we finally get

$$(4.40) \quad v_n^{*,\gamma}(y) = \sum_{\xi \in \mathbb{Z}^{d-1} \times \{0\}} c_\xi^\gamma \left(T_0^t n; \frac{y \cdot n}{n \cdot \nu_0} \right) e^{2\pi i (T_0^{-1})^t \xi \cdot y}, \quad y \in \overline{\Omega_n},$$

for the solution of (2.3).

Clearly, the entire analysis remains valid for irrational directions n satisfying $n \cdot \nu_0 < 0$.

Proof of Theorem 1.3. For $\tau > 0$ set $S_{\tau,+} = \{n \in \mathbb{S}^{d-1} : n \notin \mathbb{R}\mathbb{Q}^d, n \cdot \nu_0 > \tau\}$, and for $\xi \in \mathbb{Z}^d$ consider the function $v_\xi(y) := e^{2\pi i \xi \cdot y} I_N$, $y \in \mathbb{R}^d$, where $I_N \in M_N(\mathbb{R})$ is the identity matrix.

Let $n \in S_{\tau,+}$, and consider a boundary layer system (2.1) set on Ω_n and with boundary data v_ξ . Let $v_\xi^\infty \in M_N(\mathbb{R})$ be the corresponding constant field given by Theorem 2.1. The formula (2.5) for v_ξ^∞ in view of (2.7) and (2.8) is reduced to

$$(4.41) \quad v_\xi^\infty(n) = \int_{\partial\Omega_n} \partial_{y_\alpha} G^0(n, y) d\sigma(y) \times [c_{-\xi}(A^{\beta\alpha}) n_\beta + c_{-\xi}(\partial_{y_\beta}(\chi^{*,\alpha})^t A^{\beta\gamma}) n_\gamma + \mathcal{M}\{\partial_{y_\beta}(v_n^{*,\alpha})^t e^{2\pi i \xi \cdot y} A^{\beta\gamma}\} n_\gamma],$$

where c_ξ is the ξ -th Fourier coefficient, and $v_n^{*,\alpha}$ is the solution to (2.3). Indeed, $A^{\beta\alpha}$ is an $N \times N$ matrix for each α, β , hence $A^{\beta\alpha} v_\xi = A^{\beta\alpha} e^{2\pi i \xi \cdot y}$. Consequently, the Fourier spectrum of $A^{\beta\alpha} v_\xi$ equals the Fourier spectrum of $A^{\beta\alpha}$ shifted by $-\xi \in \mathbb{Z}^d$, in particular we get $c_0(A^{\beta\alpha}(y) e^{2\pi i \xi \cdot y}) = c_{-\xi}(A^{\beta\alpha})$. The same argument applies to the second term in the brackets in (4.41), hence the reduction of (2.5) to (4.41) follows.

To treat the term in (4.41) involving boundary layer corrector we will apply Corollary 2.5. Since $v_n^{*,\alpha}$ is smooth up to the boundary of Ω_n and has expansion (4.40) it follows that $\partial_{y_\beta}(v_n^{*,\alpha})$ has a similar expansion into exponentials obtained from term by term differentiation of the series in (4.40). Thus, if $\mathfrak{a}_n^{\alpha\beta}(\eta; t)$, for $t \geq 0$ and $\eta \in \mathbb{Z}^d$, denotes the η -th coefficient of $\partial_{y_\beta}(v_n^{*,\alpha})$ for $t = y \cdot n$, we get

$$(4.42) \quad \mathfrak{a}_n^{\alpha\beta}(\eta; y \cdot n) = \frac{n_\beta}{n \cdot \nu_0} (\partial_t c_\eta^\alpha) \left(T_0^t n; \frac{y \cdot n}{n \cdot \nu_0} \right) + 2\pi i (T_0^{-1})^t \eta \cdot e_\beta c_\eta^\alpha \left(T_0^t n; \frac{y \cdot n}{n \cdot \nu_0} \right),$$

for all $y \in \overline{\Omega_n}$, where e_β is the β -th vector in the standard basis of \mathbb{R}^d . But recall, that \mathcal{M} -averages are understood for restrictions of functions on the boundary of halfspace Ω_n (see the discussion after (2.5)). Hence, in order to be able to apply Corollary 2.5 for the set of parameters $S_{\tau,+}$ we need only to consider dependence of (4.42) on n for $y \cdot n = 0$.

Observe that from (4.40) we have $\mathfrak{a}_n^{\alpha\beta}(\eta; t) \equiv 0$ for any $\eta \in \mathbb{Z}^d$ with $\eta_d \neq 0$. Thanks to (4.39) we get

$$(4.43) \quad \mathfrak{a}_n^{\alpha\beta}(\eta; 0) = \frac{n_\beta}{n \cdot \nu_0} (\partial_t c_\eta^\alpha)(T_0^t n; 0) + 2\pi i (T_0^{-1})^t \eta \cdot e_\beta c_\eta(-\chi^{*,\alpha}),$$

thus we need to check the following two conditions in order to apply Corollary 2.5, namely for any $1 \leq \alpha, \beta \leq d$ we must have

$$(4.44) \quad \sum_{\eta \in \mathbb{Z}^{d-1} \times \{0\}} |\mathbf{a}_n^{\alpha\beta}(\eta; 0)| < \infty \text{ for all } n \in S_{\tau,+},$$

$$(4.45) \quad \sup_{\eta \in \mathbb{Z}^{d-1} \times \{0\}} |\mathbf{a}_{n^{(1)}}^{\alpha\beta}(\eta; 0) - \mathbf{a}_{n^{(2)}}^{\alpha\beta}(\eta; 0)| \leq C_0 |n^{(1)} - n^{(2)}| \text{ for any } n^{(1)}, n^{(2)} \in S_{\tau,+}.$$

Due to the choice of T_0 we have $T_0^t n \cdot e_d = n \cdot T_0 e_d = n \cdot \nu_0 > \tau$ for all $n \in S_{\tau,+}$. Using this we apply Lemma 4.8 part (a) and from (4.43) and the smoothness of $\chi^{*,\alpha}$ - the solution to cell-problem, we obtain (4.44). Next, by Lemma 4.8 part (b) and (4.43) we arrive at (4.45).

Applying Corollary 2.5 with the smooth function $e^{2\pi i \xi \cdot y} A^{\beta\gamma}$ and for each parameter $n \in S_{\tau,+}$ choosing the set of coefficients $\{\mathbf{a}_n^{\alpha\beta}(\eta; 0)\}_{\eta \in \mathbb{Z}^{d-1} \times \{0\}}$ (as the coefficients of the expansion for functions in Corollary 2.5), we obtain that the mapping $n \mapsto \mathcal{M}\{\partial_{y_\beta} (v_n^{*,\alpha})^t e^{2\pi i \xi \cdot y} A^{\beta\gamma}\}$ is Lipschitz continuous on $S_{\tau,+}$ with Lipschitz constant bounded by a constant $C_\tau = C(\tau, A, d)$, independent of n and ξ . Finally, combining this with Lemma 3.3, from (4.41) we get

$$(4.46) \quad |v_\xi^\infty(n) - v_\xi^\infty(\nu)| \leq C_\tau |n - \nu|, \quad n, \nu \in S_{\tau,+},$$

where C_τ is independent of ξ .

For $\tau > 0$ define $D_{\tau,+} = \{x \in \partial D : n(x) \notin \mathbb{RQ}^d, n(x) \cdot \nu_0 > \tau\}$, where $n(x)$ is the normal inward vector of ∂D at x . Following the notation of Section 2, for any $x, y \in \partial D_{\tau,+}$ by (2.2) we have

$$(4.47) \quad |g^*(x) - g^*(y)| \leq \sum_{\xi \in \mathbb{Z}^d} |g_\xi^*(x) - g_\xi^*(y)| \leq \sum_{\xi \in \mathbb{Z}^d} |c_\xi(x)| \times |v_\xi^\infty(n(x)) - v_\xi^\infty(n(y))| + \sum_{\xi \in \mathbb{Z}^d} |v_\xi^\infty(n(x))| \times |c_\xi(x) - c_\xi(y)| =: \Sigma_1 + \Sigma_2.$$

Recall that $c_\xi(x) = \int_{\mathbb{T}^d} g(x, z) e^{-2\pi i \xi \cdot z} dz$, where $\xi \in \mathbb{Z}^d$, $x \in \partial D$. Fix a non-zero $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{Z}^d$ and let $|\xi_\alpha| = \max_{1 \leq \beta \leq d} |\xi_\beta|$. Let also $\partial_{2,\alpha}^{d+1}$ be the partial differentiation operator acting on $g(x, \cdot)$ ($d+1$)-times in the α -th coordinate. Using the smoothness of g , from the definition of $c_\xi(x)$ we get

$$c_\xi(x) = \frac{1}{(-2\pi i \xi_\alpha)^{d+1}} \int_{\mathbb{T}^d} \partial_{2,\alpha}^{d+1} g(x, z) e^{-2\pi i \xi \cdot z} dz.$$

Combining this with a uniform bound on $|c_0(\cdot)|$ we get

$$(4.48) \quad |c_\xi(x)| \lesssim_g (1 + |\xi|)^{-(d+1)} \text{ uniformly in } x \in \partial D \text{ and } \xi \in \mathbb{Z}^d.$$

As g is smooth with respect to both of its variables, in a similar way we obtain

$$(4.49) \quad |c_\xi(x) - c_\xi(y)| \lesssim_g (1 + |\xi|)^{-(d+1)} |x - y|,$$

for all $x, y \in \partial D$ and non-zero $\xi \in \mathbb{Z}^d$. Using (4.48) and (4.46) for any $x, y \in \partial D_{\tau,+}$ we get $\Sigma_1 \leq C_{g,\tau} |n(x) - n(y)| \leq C_{g,\tau} |x - y|$, where we have used the smoothness of ∂D to obtain the second inequality. In a similar vein, in Σ_2 using Lemma 2.6 to bound the constant field and employing (4.49) leads to $\Sigma_2 \leq C_g |x - y|$. The estimates for Σ_1 and Σ_2 applied to (4.47) show that g^* is Lipschitz on $\partial D_{\tau,+}$.

Obviously, the same argument works for the other hemisphere $S_{\tau,-} = \{n \in \mathbb{S}^{d-1} : n \notin \mathbb{RQ}^d, n \cdot \nu_0 < -\tau\}$ as well. The proof of the Theorem is now completed. \square

4.5. Concluding remarks. Results concerning regularity of boundary layer tails are very few in the literature. In the same setting as we have here, namely second order divergence type elliptic systems, the smoothness of g^* under restrictive condition (1.3) on the coefficients and for $d \geq 3$, was established by H. Shahgholian, P. Sjölin, and the current author in [2] as an outcome of methods of [2] and [14] (see formula (4.4) in [2], and the discussion after that). It is easy to see that we recover this result for g^* from the proof of Theorem 1.3 above. Namely, the condition (1.3) implies that solutions to cell-problem (2.4), and hence to boundary layer systems (2.3), are trivial. This in its turn shows that in formula (4.41) the last average is vanishing, and we get that the boundary layer tail, as a function of normal n , equals to a C^∞ function almost everywhere on the sphere. The rest of the proof proceeds with minor modifications. In dimension two, the smoothness of g^* is new, while for $d \geq 3$ we get an alternative proof of the mentioned result from [2].

Concerning other settings, the reader may consult a recent work by Feldman and Kim [8], and the references therein, where they analyse continuity properties of boundary layer tails associated with fully nonlinear uniformly elliptic equations of second order.

Getting back to our case, one can see from the analysis above that the main obstacle towards the regularity of g^* comes from boundary layer correctors, in particular we do not know if the behaviour of boundary layer tails $v^\infty(n)$ is in any sense uniform with respect to normals n . A specific instance of this non-uniformity is the convergence speed of boundary layer correctors to their corresponding tails away from the boundary. Concerning this aspect in [1] we show that given any one-to-one, continuous function decreasing to 0 at infinity (i.e. a convergence rate), one may construct a problem of form (2.1) with smooth data, so that convergence towards boundary layer tail is slower than the given rate in advance. This in particular indicates that approaches toward regularity of g^* based on controlling the speed of convergence of the tails, are unlikely to lead to a positive conclusion.

It is also interesting to observe (in the light of Example 4.1) that condition (A5) implies that the operator only “sees” Diophantine directions on the hemispheres considered in the proof of Theorem 1.3. It thus leads to an idea that one may try to tailor the Diophantine condition of [9] to the given operator. Developing this line it seems plausible that one should be able to deduce the claim of Theorem 1.2 (although without any structural results such as expansion (4.40)) using instead methods of [9] combined with some of the ideas considered here, in particular Lemma A.1 and the proof of Theorem 1.3. In this perspective the approach of Section 4 should be seen as a more transparent alternative to some of the methods of [9] under condition (A5), and it will be interesting to see if the ideas considered here can be developed to lead to an actual homogenization of the problem (1.1)-(1.2) under conditions (A1)-(A5).

APPENDIX A. ON THE CHOICE OF TRANSFORMATION MATRICES

This appendix contains two results concerning the choice of transformation matrices used in Section 3 and subsection 4.4, which can be useful in further refining and extending the analysis of the present paper.

A.1. Smooth rotations. Here we analyse the choice of orthogonal matrices M sending e_d to $n \in \mathbb{S}^{d-1}$ considered in Section 3. The main purpose is to show, in a constructive fashion, that in a neighbourhood of a given $n \in \mathbb{S}^{d-1}$ there is a possibility of selection of these matrices varying smoothly with n . Interestingly such a smooth selection globally on \mathbb{S}^{d-1} in general dimensions is not available due to topological obstructions discussed below. Availability of such a choice can be used, for example, in the analysis of regularity

with respect to normals n of Green's matrices $G^{0,n}$ studied in Section 3 (see Claim 3.1 for the change in the coefficient field introduced by M).

Recall that for each $n \in \mathbb{S}^{d-1}$ we choose a matrix $M \in O(d)$ such that $Me_d = n$. Such M is clearly of the form $M = [N|n]$, where N is $d \times (d-1)$ matrix with the property that its columns form an orthonormal basis in the tangent space of \mathbb{S}^{d-1} at the point n , in particular M is defined modulo group $O(d-1)$. From this we see that the existence of orthogonal matrices M sending e_d to n and varying smoothly with n is equivalent to existence of a family of smooth vector fields $\{\mathbf{v}_1(n), \dots, \mathbf{v}_{d-1}(n)\}_{n \in \mathbb{S}^{d-1}}$ that will form an orthonormal basis in the tangent space of \mathbb{S}^{d-1} at any point n . The existence of the desired vector fields, however, is false in general⁷. Let us very briefly give some details and background on this matter.

A C^∞ -manifold X of dimension $d \geq 1$ is called *parallelizable* if there exist smooth vector fields $\{\mathbf{v}_1(x), \dots, \mathbf{v}_d(x)\}_{x \in X}$, such that at each point $x \in X$ the d -tuple $\{\mathbf{v}_i(x)\}_{i=1}^d$ forms a basis in the tangent space of X at x . It is well-known that a manifold is parallelizable if and only if its tangent bundle is trivial. On the other hand the tangent bundle of the sphere \mathbb{S}^{d-1} is trivial if and only if $d = 1, 2, 4, 8$. We refer an interested reader to works by Bott, Kervaire, and Milnor [5], [15] for details and proofs. Notice, that parallelizability does not require the basis to be orthonormal, nonetheless, it follows directly that when $d \notin \{1, 2, 4, 8\}$ one cannot fix a family of orthogonal matrices, such that $Me_d = n$, and M varies smoothly with respect to n globally on \mathbb{S}^{d-1} . However, the existence of these smooth fields locally, in the neighbourhood of each point $n \in \mathbb{S}^{d-1}$ is true, for which we give an elementary, self-contained constructive proof in the next lemma.

Lemma A.1. (Smooth selection of rotations) *Fix any point $p = (p_1, \dots, p_d) \in \mathbb{S}^{d-1}$. Then, there exists an open neighbourhood $\mathcal{P} \subset \mathbb{S}^{d-1}$ of p , and an assignment $n \mapsto M_n$ from \mathcal{P} into $O(d)$ such that for all $n \in \mathcal{P}$ we have $M_n e_d = n$, and for each $1 \leq i, j \leq d$, the real-valued function $(M_n)_{ij}$ is C^∞ on \mathcal{P} .*

Proof. The proof is by induction on dimension d . Assume that $p_1 \neq 0$, and fix a neighbourhood \mathcal{P} of p on \mathbb{S}^{d-1} where $|n_1| > |p_1|/2$, for all $n \in \mathcal{P}$. Otherwise, if $p_1 = 0$ one may simply permute the coordinate system so that after the permutation the first coordinate of p is non-zero. Thus there is no loss of generality in assuming that $p_1 \neq 0$.

Let $d \geq 2$ be fixed. To a given $n = (n_1, \dots, n_d) \in \mathcal{P}$ we wish to assign a $d \times d$ matrix $X_d(n)$ of the form

$$(A.1) \quad X_d(n) = \begin{pmatrix} a_1(n) & * & \dots & * & n_1 \\ & \dots & & 1 & n_2 \\ \vdots & * & \dots & & \vdots \\ a_{d-1}(n) & 1 & & \mathbf{0} & n_{d-1} \\ 1 & 0 & \dots & 0 & n_d \end{pmatrix}$$

where the opposite main diagonal is identically one except the element on the first row; also, with the exception of the (d, d) -th element everything below the opposite main diagonal is identically zero, and the rest of the elements above that diagonal are chosen so that to have the following properties:

- (1) $X_d(n)e_d = n$,
- (2) all columns of $X_d(n)$ are pairwise orthogonal to each other,

⁷For \mathbb{R}^d , with $d \geq 3$ odd, the non-existence directly follows from Hairy Ball Theorem, which states that there is no non-vanishing continuous, let alone smooth, tangent vector field on even-dimensional spheres.

greatest common divisor. We also recall a standard notation $\mathrm{SL}(d, \mathbb{Z})$ for the *special linear group* over integers.

Claim A.2. *For any non-zero $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$ such that $[a_1, \dots, a_d] = 1$ there exists $T \in \mathrm{SL}(d, \mathbb{Z})$ satisfying $Te_d = a$.*

Proof. Before we start, observe that the condition on greatest common divisor to be 1 is necessary which trivially follows from Euclid's algorithm.

It is enough to consider the case of $a \in \mathbb{Z}^d$ having at least one of its last two coordinates non-zero. Indeed, assume the claim holds for that class of d -tuples, and take any $a \in \mathbb{Z}^d$ with $a_{d-1} = a_d = 0$ and satisfying the condition of the claim. Then, we necessarily have $d \geq 3$, and hence can fix $1 \leq i \leq d-2$ such that $a_i \neq 0$. Clearly, one of the transpositions $(i, d-1)$ or (i, d) is even. Assume the second one, and consider $\tilde{a} \in \mathbb{Z}^d$ which is obtained from a by swapping i -th coordinate with d -th, and keeping the rest unchanged. Now, if \tilde{T} is the matrix for \tilde{a} satisfying the claim, then the matrix T obtained from \tilde{T} by swapping its i -th row with d -th satisfies $\det T = \det \tilde{T} = 1$, since the transposition (i, d) was even. Thus T satisfies the claim for the original $a \in \mathbb{Z}^d$. Given this, we will only consider $a \in \mathbb{Z}^d$ satisfying $|a_{d-1}| + |a_d| > 0$.

Next, we claim that for each $d \geq 2$ and any $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$ with $|a_{d-1}| + |a_d| > 0$, and $[a_1, \dots, a_d] = 1$ there exists a matrix $T \in \mathrm{SL}(d, \mathbb{Z})$ having a as its last column and such that all elements of T above the main diagonal, except possibly on the last column, are 0. The proof of this statement proceeds by induction on d .

The case of $d = 2$ follows directly from Euclid's algorithm. Now assume the induction hypothesis holds for $d \geq 2$, and take $a \in \mathbb{Z}^{d+1}$ such that at least one of its last two coordinates is non-zero and $d+1$ coordinates of a are coprime. Obviously, the inductive hypothesis applies to $(a_1, \dots, a_{d-1}, [a_d, a_{d+1}]) \in \mathbb{Z}^d$ and we let $T^{(0)}$ be the corresponding $d \times d$ matrix. By Euclid's algorithm there are $x, y \in \mathbb{Z}$ such that $[a_d, a_{d+1}] = xa_d + ya_{d+1}$. Clearly $[x, y] = 1$. Now, consider a $(d+1) \times (d+1)$ matrix T of the form

$$(A.3) \quad T = \begin{pmatrix} & & & & 0 & a_1 \\ & & & & \vdots & \vdots \\ & & & & 0 & a_{d-1} \\ & & T_{d-1, d-1}^{(0)} & & & \\ t_{d1} & t_{d2} & \dots & t_{dd-1} & y & a_d \\ t_{d+11} & t_{d+12} & \dots & t_{d+1d-1} & -x & a_{d+1} \end{pmatrix},$$

where $T_{d-1, d-1}^{(0)}$ is the submatrix of $T^{(0)}$ on the first $d-1$ rows and columns, and $t_{ij} \in \mathbb{Z}$, $i = d, d+1$ and $1 \leq j \leq d-1$, are parameters to be chosen below. To complete the induction step, and hence the proof of the entire claim, it remains to show that there is a choice of t_{ij} ensuring $\det T = 1$. Expanding the determinant of T with respect to its last two rows (using standard extension of the Laplace expansion) we get

$$(A.4) \quad \det T = \sum_{1 \leq i < j \leq d+1} \varepsilon_{ij} \det M_{ij} \det \tilde{M}_{ij},$$

where $\varepsilon_{ij} = (-1)^{d+(d+1)+i+j}$, M_{ij} is the 2×2 submatrix of T from elements on the rows $d, d+1$ and columns i, j , and \tilde{M}_{ij} is the $(d-1) \times (d-1)$ submatrix complementary to M_{ij} . Due to construction, \tilde{M}_{ij} contains a zero-column unless i or j in (A.4) equals d . Hence, (A.4) reduces to

$$(A.5) \quad \det T = \sum_{i=1}^{d-1} \varepsilon_{id} \det \begin{pmatrix} t_{di} & y \\ t_{d+1i} & -x \end{pmatrix} \det \tilde{M}_{id} + [a_d, a_{d+1}] \det \tilde{M}_{dd+1}$$

For each $1 \leq i \leq d-1$ we choose $t_{di}, t_{d+1i} \in \mathbb{Z}$ in order to have $\varepsilon_{id}(-t_{di}x - yt_{d+1i}) = (-1)^{i+d}T_{di}^{(0)}$, where $T_{di}^{(0)}$ is the (d, i) -th element of $T^{(0)}$. The existence of this choice follows from equality $\mathbb{Z} = x\mathbb{Z} + y\mathbb{Z}$, which on its turn is a direct consequence of Euclid's algorithm and the fact that x, y are coprime. Now with this choice of parameters t_{ij} , (A.5) coincides with the expansion of $\det T^{(0)}$ with respect to its last row, and hence equals 1. The proof of the claim is complete. \square

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