ON THE GREEDY ALGORITHM BY THE HAAR SYSTEM

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Abstract. The paper studies uniform and almost everywhere convergence of the greedy algorithm with respect to Haar system. We obtain necessary and sufficient conditions for renormalization of the Haar system that guarantee uniform convergence for functions from $C[0,1]$ and almost everywhere convergence for functions from $L^1[0,1]$.

1. Introduction

Let $\Phi = \{\varphi_n\}_{n=1}^{\infty}$ be a basis in Banach space X satisfying $\inf_n ||\varphi_n||_{\mathrm{X}} > 0$. Then any element $f \in \mathrm{X}$ can be uniquely represented by a series with respect to the system Φ which converges to f in the norm of X:

$$f = \sum_{n=1}^{\infty} c_n(f)\varphi_n,$$

where $c_n(f)$, $n = 1, 2, ...$ are the coefficients of the expansion and $\lim_{n\to\infty} c_n(f) = 0$. For $N \in \mathbb{N}$ let $\Lambda_N \subset \mathbb{N}$ be so that

$$\min_{k\in\Lambda_N} |c_k(f)| \geq \max_{k\notin\Lambda_N} |c_k(f)|.$$

Then

$$G_N(f) := G_N(f, \Phi) := \sum_{k\in\Lambda_N} c_k(f)\varphi_k$$

is said to be N-th *greedy approximant* of an element f with respect to system Φ, and the method of approximating f by a sequence G_N, $N = 1, 2, ...$ is called *greedy approximation.*

A basis Φ is called *quasi-greedy* if there exists a constant C such that for any $f \in \mathrm{X}$ one has

$$||G_N(f)||_{\mathrm{X}} \leq C||f||_{\mathrm{X}}, \qquad N = 1, 2, ...$$

In [14] P. Wojtaszczyk proved that a basis is quasi-greedy if and only if

$$\lim_{N\to\infty} ||f - G_N(f)||_{\mathrm{X}} = 0, \qquad \forall f \in \mathrm{X}.$$

The convergence of greedy algorithm for special systems was studied by many authors. Below is a list of some of the many results obtained for certain classical systems. Answering a question raised by L. Carleson and R. Coifman, T. Körner [9] constructed a function from $L^2(\mathbb{T})$, and then, in [10], a continuous function possessing almost everywhere divergent greedy algorithms with respect to trigonometric system. In [13] V. Temlyakov proved the existence of a function from $L^p(\mathbb{T})$ for all $1 \leq p < 2$, such that the greedy algorithm in trigonometric system is divergent in measure, and also the existence of a continuous function with divergent greedy algorithm in L^p norm for any $p > 2$ again with respect to trigonometric system. On the other hand S. Konyagin and V. Temlyakov [7] obtained sufficient conditions for the convergence of the greedy algorithm. Similar

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results concerning convergence and divergence of the greedy algorithm by the classical Walsh system were obtained by G. Amirkhanyan [1].

An example of a continuous function, which has divergent in measure greedy algorithm by the Faber-Schauder system is constructed by M. Grigoryan and A. Sargsyan [2].

S. Kostyukovsky and A. Olevskii [8] constructed an orthonormal basis for $L^2(0,1)$, consisting of uniformly bounded functions, such that for any function from $L^2(0,1)$ the greedy algorithm with respect to that basis converges almost everywhere. M. Nielsen [11] constructed a uniformly bounded orthonormal system which is a quasi-greedy basis in $L^p(0,1)$ for all $1<p<\infty$.

The purpose of this note is to study convergence properties of the greedy algorithm with respect to Haar system, and in particular, the interplay between the behavior of the method of approximation and renormalization of the system. Throughout the paper $\Gamma=\{\gamma_n\}_{n=1}^{\infty}$ stands for a sequence of decreasing positive numbers. For $f\in \mathrm{X}$ consider the Γ-weighted decreasing rearrangement of non-zero coefficients of f, namely let $\sigma:\mathbb{N}\to\mathbb{N}$ be a permutation of the spectrum of f such that

$$|\gamma_{\sigma(1)}c_{\sigma(1)}(f)| \geq |\gamma_{\sigma(2)}c_{\sigma(2)}(f)| \geq ... \geq |\gamma_{\sigma(n)}c_{\sigma(n)}(f)| \geq ... \tag{1.1}$$

and set

$$G_N(f,\Psi,\sigma) := \sum_{n=1}^{N} c_{\sigma(n)}(f)\varphi_{\sigma(n)}, \qquad N=1,2,... \tag{1.2}$$

Denote by $D(f,\Psi,\Gamma)$ the set of all rearrangements σ satisfying (1.1). It is easy to see that (1.2) coincides with the greedy approximant by the renormed system Ψ, namely

$$G_N(f,\Psi,\sigma) := G_N\left(f, \left\{\frac{1}{\gamma_n}\varphi_n\right\}\right), \qquad N=1,2,... \tag{1.3}$$

In [6] S. Konyagin and V. Temlyakov proved that for any normed basis Φ of Banach space X and for $\Gamma=\{2^{-n}\}_{n=1}^{\infty}$ one has

$$\lim_{N\to\infty} ||G_N(f,\Phi,\Gamma)-f||_{\mathrm{X}} = 0, \qquad \text{for any } f\in \mathrm{X}.$$

Everywhere below $\mathrm{H}_p := \{h_{n,p}\}_{n=1}^{\infty}$ is the Haar system normed in $L^p(0,1)$, $1\leq p\leq\infty$ and $\Gamma_p := \{\gamma_{n,p}\}_{n=1}^{\infty}$ where $\gamma_{n,p}=2^{-k/p}$ if $n=2^k+i$, $i=1,2,...,2^k$.

Remark 1.1. *Suppose $f\in L^1(0,1)$ and $\{c_n(f)\}_{n=1}^{\infty}$ are the coefficients of the expansion of f by the system H_∞. If $f\in C[0,1]$, then $c_n(f)\to 0$ and the rearrangement σ satisfying (1.1) exists. But the condition $f\in L^1(0,1)$ does not guarantee that $c_n(f)\to 0$, i.e. f and Γ can be such that there is no rearrangement satisfying (1.1). In such a case, the coefficients are partitioned into the following two parts:*

$$A := \{n\in\mathbb{N}: \ |c_n(f)|\leq 1\} \qquad \text{and} \qquad B := \mathbb{N}\setminus A.$$

After that the convergence of greedy approximants is understood for the rearrangements σ, such that

$$|\gamma_{\sigma(n)}c_{\sigma(n)}(f)| \geq |\gamma_{\sigma(n+1)}c_{\sigma(n+1)}(f)|, \qquad n\in A, \tag{1.4}$$

and the part B is rearranged in an arbitrary fashion.

Observe that a rearrangement satisfying (1.4) always exists if Γ tends to zero, and if it does not tend to zero, then only functions with rearrangements satisfying (1.4) are considered, and this does not lead to a loss of generality as will be seen from the analysis below.

Hereinafter, we understand the sets $D(f, \Phi, \Gamma)$ and the approximants (1.2) with regard to Remark 1.1. The following theorem due to T. Tao [12] is given in a reformulated, but equivalent form.

Theorem 1.2. *The following assertions hold true:*

a) *if* $1 < p < \infty$ *then for any* $f \in L^p(0,1)$

$$\lim_{N\to\infty} G_N(f, \mathrm{H}_\infty, \Gamma_p)(x) = f(x), \qquad a.e. \text{ in } [0,1],$$

b) *there exists a function* $f \in \bigcap\limits_{1<p<\infty} L^p(0,1)$ *such that*

$$\lim_{N\to\infty} \sup |G_N(f, \mathrm{H}_\infty, \Gamma_\infty)(x)| = \infty, \qquad \text{for any } x \in [0,1].$$

Note that by (1.2) and (1.3) we have $G_N(f, \mathrm{H}_\infty, \Gamma_p) \equiv G_N(f, \mathrm{H}_p)$ for all $1 \le p \le \infty$. For a sequence $\Gamma = \{\gamma_n\}_{n=1}^\infty$ we set

$$\tau(\Gamma) = \sup_{m>n} \left\{ \frac{m}{n} : \frac{\gamma_n}{\gamma_m} \le 2 \right\}.$$

Example 1.3. *For any* $p > 0$ *one has* $\tau(\{n^{-p}\}_{n=1}^\infty) < \infty$*, whereas* $\tau(\{(\ln n)^{-1}\}_{n=2}^\infty) = \infty$.

Remark 1.4. *It is easy to see that if* $\tau(\Gamma) < \infty$ *then* $\gamma_n \to 0$*. On the other hand merely decreasing to 0, even at a very high speed, does not guarantee that* $\tau(\Gamma) < \infty$ *as can be seen from the following example. Take any decreasing sequence* $\gamma_n > 0$ *that approaches to 0. Then it is easy to see that the sequence*

$$\Gamma = \left\{ \left\{ \left(\frac{1}{2} + \frac{1}{i} \right) \gamma_k \right\}_{i=2}^{k} \right\}_{k=2}^{\infty}$$

is decreasing, tends to 0, but $\tau(\Gamma) = \infty$.

S. Gogyan [3] proved that the system H_1 is a quasi-greedy basis in the space $L^1(0,1)$ if and only if $\tau(\Gamma) < \infty$. The following are the main results of the present note.

Theorem A. *Let* $\Gamma = \{\gamma_n\}_{n=1}^\infty$ *be a fixed sequence of weights. Then*

1) *if* $\tau(\Gamma) < \infty$ *then*

$$\lim_{N\to\infty} ||f - G_N(f, \mathrm{H}_\infty, \Gamma)||_{C[0,1]} = 0$$

for any $f \in C[0,1]$ *and any rearrangement* $\sigma \in D(f, \mathrm{H}_\infty, \Gamma)$,

2) *if* $\tau(\Gamma) = \infty$*, then there exits a function* $f \in C[0,1]$ *such that*

$$\limsup_{N\to\infty} |G_N(f, \mathrm{H}_\infty, \Gamma)(x)| = \infty, \qquad a.e. \text{ in } [0,1]$$

for any rearrangement $\sigma \in D(f, \mathrm{H}_\infty, \Gamma)$.

Theorem B. *For any $f \in L^1(0,1)$ and any rearrangement $\sigma \in D(f, \mathrm{H}_\infty, \Gamma)$ one has $\lim\limits_{N\to\infty} G_N(f, \mathrm{H}_\infty, \Gamma)(x) = f(x)$ a.e. in $[0,1]$ if and only if $\tau(\Gamma) < \infty$.*

Note that the aforementioned theorem of Tao follows from Theorems A and B, since $\tau(\Gamma_p) < \infty$ for any $1 < p < \infty$ and $\tau(\Gamma_\infty) = \infty$.

2. Notation and Auxiliary Results

Intervals of the form $\left(\frac{i-1}{2^k}, \frac{i}{2^k}\right)$, where $i = 1, 2, ..., 2^k$ and $i = 0, 1, ...$ are called dyadic intervals. For any integer $n = 2^k + i$ with $k \geq 0$ and $1 \leq i \leq 2^k$ we set

$$\Delta_n = \Delta_k^i = \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right), \qquad \overline{\Delta}_n = \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right],$$

$$\Delta_1 = \Delta_0^0 = (0,1), \qquad \overline{\Delta}_0 = [0,1].$$

Denote by Δ_n^+ and Δ_n^- correspondingly the left and right halves of dyadic intervals:

$$\Delta_n^+ = (\Delta_k^i)^+ = \left(\frac{i-1}{2^k}, \frac{2i-1}{2^{k+1}}\right) = \Delta_{k+1}^{2i-1},$$

$$\Delta_n^- = (\Delta_k^i)^- = \left(\frac{2i-1}{2^{k+1}}, \frac{i}{2^k}\right) = \Delta_{k+1}^{2i}.$$

We denote the midpoint of the interval Δ_n by t_n; clearly for $n = 2^k + i$ with the usual convention, we have $t_n = \frac{2i-1}{2^{k+1}}$.

By $\mathrm{H}_\infty = \{h_n\}_{n=1}^\infty$ we denote the Haar system normalized in $||\cdot||_\infty$ norm, and for a function $f \in L^1(0,1)$ we denote by $c_n(f)$ the n-th coefficient of f with respect to H_∞. For any rearrangement $\sigma \in D(f, \mathrm{H}_\infty, \Gamma)$ we define the set

$$\Lambda_N(f) := \Lambda_N(f, \sigma) := \{\sigma(1), ..., \sigma(N)\}, \qquad N = 1, 2, ...$$

and introduce

$$G_N^*(f, x) := G_N^*(f, \sigma, x) := \sup_{1\leq k\leq N} \left| \sum_{k=1}^N c_{\sigma(k)}(f) h_{\sigma(k)}(x) \right|, \; x \in [0,1], \; N = 1, 2, ...,$$

for the majorant of greedy operators. Also, we set

$$\mathrm{sp} f := \{n \in \mathbb{N} : \; c_n(f) \neq 0\},$$

The symbol $a \asymp b$ means a double inequality of the form $C_1 a \leq b \leq C_2 b$, where $C_1, C_2 > 0$ are absolute constants. Throughout the text by C we denote an absolute constant which may be different in different formulas.

We recall some well-known facts about Haar system which we will need later.

Lemma 2.1. *(see [4], page 79) Let $A_0 = \{(0,1), \emptyset\}$, $A_j = \{(0,1), \emptyset, \Delta_n^+, \Delta_n^- : \; n = 1, 2, ..., j\}$ and let $\mathfrak{F}_j$ be the family of sets that can be represented as a finite union of intervals from A_j, $j = 0, 1, ...$. Further, let $\alpha(x)$, $x \in [0,1]$ be a function that takes values from the set $\mathbb{Z}_+ \cup \{\infty\}$ and is such that*

$$e_j := \{x \in (0,1) : \; \alpha(x) = j\} \in \mathfrak{F}_j, \qquad j = 0, 1, ...$$

Then there exists a sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\varepsilon_n \in \{0,1\}$ for all $n \in \mathbb{N}$ and satisfying

$$\varepsilon_n h_n(x) = \begin{cases} h_n(x), & \text{for } n \leq \alpha(x), \\ 0, & \text{for } n > \alpha(x), \end{cases}$$

for all $n \geq 1$ *and* $x \in [0,1] \setminus R_2$*, where* $R_2 = \{\{i/2^k\}_{i=1}^{2^k}\}_{k=0}^{\infty}$.

As a consequence, one gets that for any real numbers c_n*,* $n = 1, 2, ...$ *the following is true*

$$\sum_{n=1}^{\alpha(x)} c_n h_n(x) = \sum_{n=1}^{\infty} \varepsilon_n c_n h_n(x), \qquad x \in (0,1) \setminus R_2,$$

with the convention that $\sum\limits_{n=1}^{0} \equiv 0$.

Lemma 2.2. *(see* [4]*, page 93) For any polynomial of the form*

$$\sum_{n=M}^{N} a_n h_n(x), \qquad 1 < M < N,$$

there is a rearrangement $\{\sigma(n)\}_{n=M}^{N}$ *of the numbers* $M, M+1, ..., N$ *such that*

$$\max_{M \leq p \leq q \leq N} \left| \sum_{n=p}^{q} a_{\sigma(n)} h_{\sigma(n)}(x) \right| \geq \frac{1}{4} \sum_{n=p}^{q} |a_n h_n(x)|, \qquad x \in [0,1].$$

Remark 2.3. *It follows from the proof of Lemma 2.2, that if the all coefficients* a_n *are of the same sign, then the rearrangement* σ *can be chosen from the condition*

$$0 < t_{\sigma(M)} < t_{\sigma(M+1)} < ... < t_{\sigma(N)} < 1, \tag{2.1}$$

where t_n *is the midpoint of the interval* Δ_n *and the constant* $1/4$ *can be replaced by* $1/2$.

We will also need the following result.

Theorem 2.4. *(see* [4]*, page 87) The Haar series*

$$\sum_{n=1}^{\infty} a_n h_n(x)$$

converges a.e. in a set $E \subset (0,1)$*, where* $\mu(E) > 0$*, if and only if*

$$\sum_{n=1}^{\infty} a_n^2 h_n^2(x) < \infty, \qquad a.e.\ in\ E.$$

Let us prove the following Lemma.

Lemma 2.5. *Let* $\Gamma = \{\gamma_n\}_{n=1}^{\infty}$ *be a decreasing sequence of positive numbers, and let positive integers* $m, p > 1$ *and* $A < B$ *be such that*

$$[2^{p+1}, 2^{p+m^7+1}] \subset [A, B] \qquad \text{and} \qquad \frac{\gamma_A}{\gamma_B} \leq 2. \tag{2.2}$$

Then, for any sequence $\{\xi_i\}_{i=1}^{\infty}$*,* $|\xi_i| \leq m^{-8}$*,* $i = 1, 2, ...$*, there exists a function* $f \in C[0,1]$ *and a set* $L \subset \mathrm{sp} f$ *such that*

1. $||f||_{C[0,1]} \leq 3$,
2. $c_j(f) = 0$ *for any* $j \leq p$,
3. $\frac{1}{4m^4} < c_j(f) < \frac{4}{m^4}$ *for any* $i, j \in L$,

4. *if* $\overline{f} := f + \sum_{n\in L} \xi_n h_n$ *and* $\sigma \in D(\overline{f}, \mathrm{H}_\infty, \Gamma)$, *then the inclusion* $\Lambda_q(\overline{f}) \supset L$ *implies the inequality*

$$\mu\left\{x \in [0,1]:\ G_q^*(\overline{f}, x) \geq \frac{1}{12}m^3\right\} \geq 1 - \frac{C}{m}.$$

Proof. Consider the following polynomial

$$P(x) = \frac{1}{m^4}\sum_{k=p+1}^{p+m^7}\sum_{i=1}^{2^k}\tau_k^i h_k^i(x), \tag{2.3}$$

where $\tau_k^i \in [1/3, 3]$ and will be chosen in a moment. Obviously,

$$\frac{1}{m^4}\sum_{k=p+1}^{p+m^7}\sum_{i=1}^{2^k}\tau_k^i |h_k^i(x)| \asymp m^3 \text{ a.e. in } [0,1] \text{ and } ||P||_2 \asymp \frac{1}{\sqrt{m}}, \tag{2.4}$$

independently of the choice of the numbers $\{\tau_k^i\}$. Assume that $\mathrm{sp}P = \{M_i\}_{i=1}^{\nu}$, where $M_1 < M_2 < ... < M_\nu$ and σ is a rearrangement of the set $\mathrm{sp}P$, satisfying (2.1). We choose numbers $\{\tau_k^i\}$ as follows

$$\tau_{M_i} = \frac{\gamma_{\sigma^{-1}(M_i)}}{\gamma_{M_i}}, \qquad i = 1, 2, ..., \nu. \tag{2.5}$$

Next, in view of (2.2) and monotonicity of Γ we have $1/2 \leq \tau_n \leq 2$ for any $n \in \mathrm{sp}P$, and due to the construction of rearrangement σ we get that $\sigma \in D(P, \mathrm{H}_\infty, \Gamma)$. Since $1/3 < \tau_n < 3$ for $n \in \mathrm{sp}P$, by slightly perturbing the numbers τ_n we can reach the state that all the numbers $|\gamma_n c_n(P)|$ are pairwise different for $n \in \mathrm{sp}P$, and $\tau_n \in [1/3, 3]$ and the mentioned properties of the polynomial P and the rearrangement σ are preserved. Therefore, without loss of generality we will assume that $\# D(P, \mathrm{H}_\infty, \Gamma) = 1$.

Consider the function

$$\alpha(x) = \begin{cases} \infty, & \text{if } \sup\limits_{N: x\in\Delta_N} ||S_N(P)||_{C(\Delta_N)} \leq 1, \\ \inf\{N :\ ||S_{N+1}(P)||_{C(\Delta_{N+1})} > 1,\ x \in \Delta_{N+1}\}, & \text{otherwise} , \end{cases}$$

where S_N is the N-th partial sum operator with respect to Haar system. Let $e_j := \{x \in [0,1] :\ \alpha(x) = j\}$, $j = 0, 1, 2, ...$ Then, clearly $e_j = \Delta_{j+1}$, $j = 0, 1, ...$ if $e_j \neq \emptyset$, i.e. the function $\alpha(x)$ satisfies the conditions of Lemma 2.1. Hence, there exists a sequence $\varepsilon_n \in \{0, 1\}$, $n = 1, 2, ...$ such that

$$Q(x) := \sum_{n=1}^{\alpha(x)} c_n(P)h_n(x) = \sum_{n=1}^{\infty}\varepsilon_n c_n(P)h_n(x), \qquad x \in [0,1] \setminus R_2. \tag{2.6}$$

It is clear that Q is a polynomial by Haar system and

$$||Q||_{C[0,1]} \leq 1 \qquad \text{and} \qquad ||Q||_2 \leq C\frac{1}{\sqrt{m}}.$$

Set

$$E := \{x \in [0,1] : \ \alpha(x) = \infty\}, \quad S^*(P,x) := \sup_{1 \le N < \infty} \left| \sum_{n=1}^{N} c_n(P) h_n(x) \right|, \quad x \in [0,1],$$

and recall the following estimate (see [4], p. 88)

$$||S^*(f)||_p \asymp ||f||_p, \qquad 1 < p < \infty. \tag{2.7}$$

We have $\{x \in [0,1] : \ \alpha(x) < \infty\} = \bigcup_{j=0}^{\infty} e_j$ and $e_j = \Delta_{j+1}$ when $e_j \neq \emptyset$. Then observe that the function $S_{j+1}(P,x)$ is constant on the intervals Δ^+_{j+1} and Δ^-_{j+1} and hence either Δ^+_{j+1} or Δ^-_{j+1} is contained in the set $\{x \in [0,1] : \ S^*(P,x) > 1\}$. Since the sets e_j do not overlap, we obtain

$$\mu\{x \in [0,1] : \ \alpha(x) < \infty\} \le 2\mu\{x \in [0,1] : \ S^*(P,x) > 1\}. \tag{2.8}$$

By (2.8), Chebyshev's inequality and (2.7) we get

$$\mu\{x \in [0,1] : \ \alpha(x) < \infty\} \le 2||S^*(P,x)||_2^2 \le C||P||_2^2 \le \frac{C}{m}.$$

Thus,

$$\mu(E) \ge 1 - \frac{C}{m}, \tag{2.9}$$

and according to (2.4) we obtain

$$\sum_{n=1}^{\infty} |c_n(Q) h_n(x)| = \sum_{n=1}^{\infty} |c_n(P) h_n(x)| \asymp m^3, \qquad x \in E. \tag{2.10}$$

We set $L := \mathrm{sp} Q$ and $\overline{Q}(x) := \sum_{n \in L} (c_n(Q)\xi_n) h_n(x)$ and suppose that $0 < x_1 < x_2 < \ldots < x_r < 1$ are the all discontinuity points of the polynomial $\overline{Q}$. We fix an integer $k > p + m^7 + 1$ large enough to ensure

$$0 < x_1 - \frac{1}{2^{k+1}} < x_r + \frac{1}{2^{k+1}} < 1$$

and

$$x_i + \frac{1}{2^{k+1}} < x_{i+1} - \frac{1}{2^{k+1}}, \qquad i = 1, 2, \ldots, r-1.$$

We now introduce a function f continuous on $[0,1]$ as follows

$$f(x) = \begin{cases} \overline{Q}(x), & \text{if } [0,1] \setminus \left(x_i - \frac{3}{2^{k+3}}, x_i + \frac{3}{2^{k+3}}\right) \text{ and } x = x_i \pm \frac{j}{2^{k+3}}, \ j = 1,3 \ , \\ \frac{3}{2}\overline{Q}(x_i \pm 0), & \text{if } x = x_i \pm \frac{2}{2^{k+3}} \ , \\ 0, & \text{if } x = x_i \ , \end{cases}$$

where $1 \le i \le r$ and in the remaining intervals f is extended as linear and continuous function. Clearly

$$c_n(f) = c_n(\overline{Q}), \qquad n \le 2^k. \tag{2.11}$$

Let us prove that f and L satisfy the requirements of the Lemma. By construction we have $f \in C[0,1]$ and $||f||_{C[0,1]} \le 3$, which is the first assertion of the Lemma. The second claim of the Lemma follows easily from (2.3), (2.6) and (2.11). The third assertion is due to the construction of f and the fact that $|\xi_n| \le m^{-8}$, for all $n = 1, 2, \ldots$. For the forth one define $\overline{f} := f + \sum_{n\in L} \xi_n h_n$. Then in view of (2.3), (2.6) and (2.11) one has

$$c_n(\overline{f}) = c_n(Q) = c_n(P) \qquad \text{for all } n \in L. \tag{2.12}$$

As $\#D(P, \mathrm{H}_\infty, \Gamma) = 1$ from (2.5) and (2.12) we get that for any $\sigma \in D(\overline{f}, \mathrm{H}_\infty, \Gamma)$ the set sp$P$ is rearranged according to (2.1). Combining this with (2.9), (2.10) and Lemma 2.2 we obtain the forth assertion of the current Lemma, thus finishing the proof.

□

3. Proofs of the Theorems

Proof of Theorem A. We start with the first statement of the theorem. Assume that $\tau(\Gamma) < \infty$, $f \in C[0,1]$ and $\varepsilon > 0$ is fixed. Denote

$$T_\varepsilon(f)(x) := \sum_{n: |c_n(f)\gamma_n| > \varepsilon} c_n(f) h_n(x), \qquad x \in [0,1],$$

and

$$N(\varepsilon) = \min\{N \in \mathbb{N} : \ |c_n(f)\gamma_n| \le \varepsilon, \ \forall n \ge N\}.$$

Then clearly

$$\{n \in \mathbb{N} : \ |c_n(f)\gamma_n| > \varepsilon\} \subset \{1, 2, \ldots, N(\varepsilon)\},$$

and

$$\frac{\varepsilon}{\gamma_{N(\varepsilon)}} \le |c_{N(\varepsilon)}(f)| \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.1}$$

Also, it is not hard to see that

$$||S_{N(\varepsilon)}(f) - T_\varepsilon(f)||_{C[0,1]} = \left\| \sum_{\substack{n \le N(\varepsilon) \\ |c_n(f)\gamma_n| \le \varepsilon}} c_n(f) h_n(x) \right\|_{C[0,1]} \le \varepsilon \sum_{k=0}^{[\log_2 N(\varepsilon)]-1} \frac{1}{\gamma_{n_k}} + \frac{\varepsilon}{\gamma_{N(\varepsilon)}}, \tag{3.2}$$

where $n_k = 2^{k+1}$. If $l_0 := [\log_2 \tau(\Gamma)] + 1$, then

$$\frac{n_{k+l_0}}{n_k} > \tau, \qquad k = 0, 1, \ldots, [\log_2 N(\varepsilon)] - l_0 - 1,$$

and hence

$$\frac{\gamma_{n_k}}{\gamma_{n_{k+l_0}}} > 2, \qquad k = 0, 1, \ldots, [\log_2 N(\varepsilon)] - l_0 - 1. \tag{3.3}$$

By (3.3) we obtain

$$\sum_{k=0}^{[\log_2 N(\varepsilon)]-1} \frac{1}{\gamma_{n_k}} \le \sum_{r=0}^{l_0-1} \sum_{k \equiv r (\mathrm{mod}\ l_0)} \frac{1}{\gamma_{n_k}} \le \sum_{r=0}^{l_0-1} \left(\frac{1}{2^{i_r}} + \frac{1}{2^{i_r - 1}} + \ldots + 1 \right) \frac{1}{\gamma_{N(\varepsilon)}} \le \frac{C}{\gamma_{N(\varepsilon)}}, \tag{3.4}$$

for some integers i_r, $r = 0, 1, \ldots, l_0 - 1$. By (3.2), (3.4) and (3.1) we get

$$||S_{N(\varepsilon)}(f) - T_\varepsilon(f)||_{C[0,1]} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.5}$$

As the partial sums $S_N(f)$ converge uniformly to f on $[0,1]$, from (3.5) we have

$$\lim_{\varepsilon \to 0} ||f - T_\varepsilon(f)||_{C[0,1]} = 0. \tag{3.6}$$

Observe that, if the set $D(f, \mathrm{H}_\infty, \Gamma)$ contains only one element then for any $N \in \mathbb{N}$ there exists some $\varepsilon = \varepsilon(N) > 0$ for which $G_N(f) = T_\varepsilon(f)$, implying

$$\lim_{N\to\infty} ||f - G_N(f)||_{C[0,1]} = 0.$$

In the case when $\#D(f, \mathrm{H}_\infty, \Gamma) > 1$ we set

$$\Omega_0 = \emptyset,\ \Omega_n = \{k \in \mathbb{N} \setminus (\Omega_1 \cup ... \cup \Omega_{n-1}) :\ |\gamma_k c_k(f)| = |\gamma_n c_n(f)|\},\ n = 1, 2, ..., \tag{3.7}$$

and if $\Omega_n \neq \emptyset$ we set $\omega_n := \max \Omega_n$. Note that Ω_n-s are exactly the sets due to which we get non uniqueness of the greedy rearrangement. To finish the proof it is enough to obtain a uniform control over contributions of Ω-s. Now, if $\#\Omega_n > 1$ then by the same argument as in the proof of the estimate (3.4) we obtain

$$\begin{aligned}\sum_{k\in\Omega_n} |c_k(f)h_k(x)| = |\gamma_{\omega_n} c_{\omega_n}(f)| \sum_{k\in\Omega_n} \frac{1}{\gamma_k}|h_k(x)| \leq \\ |\gamma_{\omega_n} c_{\omega_n}(f)| \sum_{k=1}^{\omega_n} \frac{1}{\gamma_k}|h_k(x)| \leq C|c_{\omega_n}(f)|, \qquad x \in [0, 1].\end{aligned} \tag{3.8}$$

The last expression combined with (3.6) completes the proof of the first statement of Theorem A.

We now proceed to the proof of the second part of Theorem A. If $\tau(\Gamma) = \infty$ then for any positive integers k and m there exists indices $B > A > m$ such that

$$\frac{B}{A} > k \qquad \text{and} \qquad \frac{\gamma_A}{\gamma_B} \leq 2. \tag{3.9}$$

By means of induction over $n = 1, 2, ...$ we construct a sequence of functions $f_n \in C[0, 1]$, sets of indices L_n, indices q_n, along with auxiliary indices p_n, A_n, B_n and sequences $\{\xi_i^n\}_{i=1}^\infty$ satisfying the following conditions:

(a) $||f_n||_{C[0,1]} \leq 3/n^2$,

(b) $\max L_n < \min L_{n+1}$ and $q_n < q_{n+1}$,

(c) $c_j(f_n) = 0$ for $j < \min L_n$,

(d) if $F_n = \sum_{k=1}^{n} f_k$, then $\max_{j\in L_{k+1}} |\gamma_j c_j(F_n)| < \min_{j\in L_k} |\gamma_j c_j(F_n)|$ for $k = 1, 2, ..., n-1$,

(e) $\mu\left\{x \in [0, 1] :\ G^*_{q_n}(F_n, \sigma, x) \geq \frac{1}{12}n\right\} \geq 1 - \frac{C}{n}$ for any $\sigma \in D(F_n, \mathrm{H}_\infty, \Gamma)$, provided that q_n is chosen so that $\Lambda_{q_n}(F_n, \sigma) \supset L_n$.

Start with $n = 1$, set $m_1 = 2n$ and choose integers $B_1 > A_1 > 1$ such that

$$\frac{B_1}{A_1} > 2^{m_1^7+3} \qquad \text{and} \qquad \frac{\gamma_{A_1}}{\gamma_{B_1}} \leq 2. \tag{3.10}$$

The possibility of such a choice follows from (3.9). We then take $p_1 = [\log_2 A_1] + 1$, $\{\xi_i^1\}_{i=1}^\infty = (0, 0, ...)$ and observe that in view of the definition of p_1 and condition (3.10) we have $[2^{p_1}, 2^{p_1+m_1^7+2}] \subset [A_1, B_1]$.

We now apply Lemma 2.5 with initial conditions m_1, p_1, A_1, B_1 and $\{\xi_i^1\}_{i=1}^\infty$ and let f_1 and L_1 be correspondingly the continuous function and the index set provided by Lemma

2.5. Set $F_1 := f_1$ and choose an integer q_1 so that $\Lambda_{q_1}(F_1) \supset L_1$. The conditions (a)-(e) are trivially satisfied.

Now assume that for $n \in \mathbb{N}$ we have constructed the quantities f_k, F_k, L_k, p_k, q_k, A_k, B_k and $\{\xi_i^k\}_{i=1}^{\infty}$ satisfying conditions (a)-(e) for all $k = 1, 2, ..., n$, and set $m_{n+1} = 2(n+1)$. Since $c_k(F_n) \to 0$ as $k \to \infty$ there exists an integer $K_0 > q_n$ such that for any $k \geq K_0$, $|c_k(F_n)| < m_{n+1}^{-10}$. Next, we fix some integers $B_{n+1} > A_{n+1}$ such that

$$A_{n+1} > \max\{B_n, 2^{K_0+2}\}, \qquad \frac{B_{n+1}}{A_{n+1}} > 2^{m_{n+1}^7+3} \qquad \text{and} \qquad \frac{\gamma_{A_{n+1}}}{\gamma_{B_{n+1}}} \leq 2. \tag{3.11}$$

Taking $p_{n+1} = [\log_2 A_{n+1}]+1$, we get by (3.11) that $[2^{p_n+1}, 2^{m_{n+1}^7+p_{n+1}+1}] \subset [A_{n+1}, B_{n+1}]$. Next we choose a sequence $\{\xi_i^{n+1}\}_{i=1}^{\infty}$ as follows

$$\xi_i^{n+1} = \begin{cases} 0, & \text{for } i < p_{n+1}, \\ m_{n+1}^2 c_i(F_n), & \text{for } i \geq p_{n+1}. \end{cases}$$

Now let g_{n+1} be a continuous function on $[0,1]$ and L_{n+1} be an index set satisfying Lemma 2.5 with the initial conditions m_{n+1}, p_{n+1}, A_{n+1}, B_{n+1} and $\{\xi_i^{n+1}\}_{i=1}^{\infty}$. Set $f_{n+1} := g_{n+1}/(n+1)^2$, $F_{n+1} := F_n + f_{n+1}$ and choose an index q_{n+1} so that $\Lambda_{q_{n+1}}(F_{n+1}) \supset L_{n+1}$. It is then easy to see that the conditions (a)-(e) are satisfied.

Let us now show that the function $f := \sum\limits_{n=1}^{\infty} f_n$ satisfies the second statement of Theorem A. Clearly $f \in C[0,1]$ by (a). Next, observe that for any rearrangement $\sigma \in D(f, \mathrm{H}_\infty, \Gamma)$ by the conditions (b)-(d) we have that the quantities $|c_n(f)\gamma_n|$ are arranged in decreasing order in any block L_n, moreover for any $m > k$ the quantities $|c_n(f)\gamma_n|$ of the block L_k are strictly larger the corresponding quantities of the block L_m. Consequently, by choosing indices q_n' so that $\Lambda_{q_n'}(f, \sigma) \supset \Lambda_n$, $n = 1, 2, ...$ by (e) we obtain

$$\mu\left\{x \in [0,1] : \ G_{q_n'}^*(f, \sigma, x) \geq \frac{1}{12} n\right\} \geq 1 - \frac{C}{n}.$$

This last inequality means that $G_N(f, \sigma, x)$ diverges to $+\infty$, finishing the proof of the Theorem.

Proof of Theorem B. The necessity obviously follows from the second statement of Theorem A, we now proceed to the proof of sufficiency. Recall that the condition $\tau(\Gamma) < \infty$ implies that $\gamma_n \searrow 0$, and if $\tau(\Gamma) < \infty$ then for $f \in L^1(0,1)$ the sets $D(f, \mathrm{H}_\infty, \Gamma)$ are understood in accordance with Remark 1.1.

Observe that the condition $f \in L^1(0,1)$ does not necessarily imply a decay of the coefficients of f, and hence in this situation we can not apply the arguments of Theorem A directly. To overcome this obstruction we set

$$A_0 = \emptyset, \quad A_n = \left\{k \in \mathbb{N} \setminus (A_1 \cup ... \cup A_{n-1}) : \ |c_k(f)| \geq \frac{1}{n}\right\}, \ n = 1, 2, ...,$$

and note that only the infinite sets A_n are of interest. Denote

$$\iota(n) = \begin{cases} 0, & \text{if } \#A_n < \infty, \\ n, & \text{if } \#A_n = \infty, \end{cases}$$

clearly for any $n \in \mathbb{N}$ the set $A_{\iota(n)}$ is either empty or a set of infinite cardinality. Applying Theorem 2.4 and in view of the fact that Fourier-Haar series of f converge to it pointwise a.e., we obtain

$$\sum_{k\in A_{\iota(n)}} |c_k(f)h_k(x)| \le n^2 \sum_{k\in A_{\iota(n)}} |c_k(f)|^2|h_k(x)|^2 < \infty \text{ a.e.}, \qquad n = 1, 2, ...,$$

that is we have absolute convergence in the blocks $A_{\iota(n)}$. We now fix some $\delta > 0$ and assume that an integer $N_0 \in \mathbb{N}$ is chosen so that $|c_k(f)| < \delta$ for any $k \in A_{\iota(n)}$ and $n \ge N_0$. We then set $B := \mathbb{N} \setminus (A_{\iota(1)} \cup ... \cup A_{\iota(N_0)})$ and

$$N(\varepsilon) := \min\{N \in B : \ |c_n(f)\gamma_n| \le \varepsilon, \ \forall n > N, \ n \in B\}, \quad \varepsilon > 0.$$

Obviously

$$\{n \in B : \ |c_n(f)\gamma_n| > \varepsilon\} \subset \{1, 2, ..., N(\varepsilon)\},$$

and

$$\frac{\varepsilon}{\gamma_{N(\varepsilon)}} \le |c_{N(\varepsilon)}| \le \delta.$$

We split both sums $T_\varepsilon(f)$ and $S_{N(\varepsilon)}(f)$ in two components as follows:

$$T_\varepsilon(f) = \sum_{n\in B:\ |c_n(f)\gamma_n|>\varepsilon} c_n(f)h_n + \Sigma_1,$$

$$S_{N(\varepsilon)}(f) = \sum_{n\in B,\ n=1}^{N(\varepsilon)} c_n(f)h_n + \Sigma_2.$$

Then

$$|T_\varepsilon(f) - S_{N(\varepsilon)}(f)| \le |\Sigma_1 - \Sigma_2| + \left| \sum_{n\in B:\ |c_n(f)\gamma_n|>\varepsilon} c_n(f)h_n - \sum_{n\in B,\ n=1}^{N(\varepsilon)} c_n(f)h_n \right|.$$

Taking into account the absolute convergence in the blocks $A_{\iota(n)}$ and that N_0 is fixed, we get that the first summand in the right hand side of the last inequality tends to zero a.e. as $\varepsilon \to 0$. As in the proof of the first statement of Theorem A on can show that the second summand in the inequality is less than a constant times $|c_{N(\varepsilon)}(f)|$. Since $|c_{N(\varepsilon)}(f)| \le \delta$ with $\delta > 0$ arbitrary small, we obtain that $T_\varepsilon(f)(x)$ converges to $f(x)$ a.e. as $\varepsilon \to 0$.

As in the proof of the convergence in Theorem A, we note that if for the function f there is only one decreasing rearrangement, then for each $N \in \mathbb{N}$ there exists some $\varepsilon = \varepsilon(N) > 0$ satisfying $G_N(f) = T_\varepsilon(f)$, which clearly implies a.e. convergence of $G_N(f)(x)$ to $f(x)$. In the case when $\#D(f, \mathrm{H}_\infty, \Gamma) > 1$ we fix $m \in \mathbb{N}$ and denote $B_m := A_{\iota(1)} \cup ... \cup A_{\iota(m)}$, $g_m := f - \sum_{n\in B_m} c_n(f)h_n(x)$, and for $n = 0, 1, ...$ define $\Omega_n(g_m)$ in accordance with (3.7). We have absolute convergence on the blocks $A_{\iota(n)}$, and hence on each B_m. Also note that non uniqueness of decreasing rearrangement is due to the sets $\Omega_n(g_m)$, however using the same argument as we had for proving (3.8), it is easy to see that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k\in\Omega_n(g_m)} |c_k(g_m)h_k(x)| = 0, \qquad x \in [0, 1].$$

This last expression together with a.e. convergence of thresholds $T_\varepsilon(f)(x)$ to $f(x)$ finishes the proof of Theorem B.

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