East Journal on Approximations

Volume 16, Number 3 (2010), 273-296

ON GREEDY ALGORITHM BY RENORMED FRANKLIN SYSTEM

HAYK ALEKSANYAN

Abstract. We characterize the all weighted greedy algorithms with respect to Franklin system which converge uniformly for continuous functions and almost everywhere for integrable functions. In case, when the algorithm fails to satisfy our classification criteria, we construct a continuous function for which the corresponding approximation diverges unboundedly almost everywhere. Some applications to wavelet systems are also discussed.

Key Words: Greedy Algorithm, Franklin system, Renormalization, Convergence

1. Introduction

Let X be a Banach space and $\Psi = \{\psi_n\}_{n=1}^{\infty}$ be a basis in X with $\inf_n ||\psi_n|| > 0$. For any $f \in X$ one has the expansion

$$f = \sum_{n=1}^{\infty} c_n(f)\psi_n,$$

where $\{c_n(f)\}$ are uniquely determined by f and $\lim_{n\to\infty} c_n(f) = 0$. Let Λ_N be a set of integers of cardinality N with

$$\min_{k\in\Lambda_N} |c_k(f)| \geq \max_{k\notin\Lambda_N} |c_k(f)|, \ N = 1, 2, ...$$

The operator

$$G_N(f) := G_N(f, \Psi) := \sum_{k\in\Lambda_N} c_k(f)\psi_k,$$

is called N-th greedy approximant of f by the system Ψ and the method of approximation of f by $G_N(f)$ is called *greedy algorithm*.

A basis Ψ is called *quasi-greedy*, if there exists a constant C such that for any f

$$||G_N(f, \Psi)|| \leq C||f||, \ N = 1, 2, ...$$

P. Wojtaszczyk [24] proved that a basis Ψ is quasi-greedy if and only if for any f the greedy algorithm converges to f, that is

$$\lim_{N\to\infty} ||f - G_N(f,\Psi)|| = 0.$$

Convergence of greedy algorithm for special systems was studied by many authors. T.W. Körner answering a question raised by Carleson and Coifman constructed in [17] a function from L^2 and then in [18] a continuous function for which the greedy algorithm by the trigonometric system diverges almost everywhere.

For trigonometric system V.Temlyakov [23] proved existence of a function from L^p, $1 \le p < 2$ whose greedy algorithm diverges in measure, and existence of a continuous function whose greedy algorithm does not converge in L^p, $p > 2$. On the other hand S. Konyagin and V. Temlyakov [16] obtained sufficient conditions for convergence of greedy algorithm. Similar results concerning convergence and divergence of greedy algorithm by the Walsh system were obtained by G. Amirkhanyan (see [2]).

M. Grigoryan and A. Sargsyan [11] constructed a continuous function for which the greedy algorithm by the Faber-Schauder system does not converge in measure.

For orthonormal Franklin system and wavelet systems with the rate of decay $\frac{C}{(1+|x|)^{2+\varepsilon}}$, for some $\varepsilon > 0$ G. Gevorkyan and A. Stepanyan [9] constructed a function from $\bigcap\limits_{1\le p<\infty} L^p(\mathbb{R})$ whose greedy algorithm diverges almost everywhere.

There are some positive results in this direction. For instance S. Kostyukovsky and A. Olevskii [20] constructed an orthonormal basis for $L^2(0,1)$ consisting of uniformly bounded functions such that the greedy algorithm for each $f \in L^2(0,1)$ by that system converges almost everywhere, and in [21] Nielsen constructed an orthonormal system of uniformly bounded functions which is a quasi-greedy basis in $L^p(0,1)$ for all $1 < p < \infty$.

For rapidly decreasing one-dimensional wavelet systems T. Tao [22] proved that the wavelet expansion of any $f \in L^p(\mathbb{R})$, $1 < p < \infty$ converges almost everywhere under the wavelet projection, hard sampling and soft sampling summation methods.

Let $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ be a decreasing sequence of positive numbers. For $f \in \mathrm{X}$ we consider the decreasing rearrangement of absolute values of non-vanishing coefficients of f with the weight γ_n:

$$|\gamma_{\sigma(1)}c_{\sigma(1)}(f)| \ge |\gamma_{\sigma(2)}c_{\sigma(2)}(f)| \ge ... \ge |\gamma_{\sigma(n)}c_{\sigma(n)}(f)| \ge ..., \tag{1.1}$$

and define the weighted greedy approximant of f as follows:

$$(1.2)\quad G_N(f,\Psi,\Gamma,\sigma) := G_N(f,\Psi,\Gamma) := \sum_{n=1}^{N} c_{\sigma(n)}(f)\psi_{\sigma(n)},\ N = 1,2,...$$

Such a weighted greedy approximation was considered in [10] (see also [13]). We denote the set of rearrangements satisfying (1.1) by $D(f,\Psi,\Gamma)$. One can see that (1.2) coincides with the greedy approximants by the renormed system Ψ:

$$(1.3)\qquad G_N(f,\Psi,\Gamma) = G_N\left(f, \left\{\frac{1}{\gamma_n}\psi_n\right\}\right).$$

In [15] S. Konyagin and V. Temlyakov proved that if $\Psi = \{\psi_n\}_{n=1}^{\infty}$ is a normed basis in a Banach space X, $\Gamma = \{2^{-n}\}_{n=1}^{\infty}$, then for any $f \in \mathrm{X}$

$$\lim_{N\to\infty} ||f - G_N(f,\Psi,\Gamma)||_X = 0.$$

We denote by $\mathrm{H}_\infty = \{h_n\}_{n=1}^{\infty}$ the Haar system and by $\mathcal{F}_\infty = \{\widetilde{f}_n\}_{n=0}^{\infty}$ the Franklin system both normed in $||\cdot||_\infty$ norm. Let Ψ be either H_∞ or $\mathcal{F}_\infty$, $f \in L^1(0,1)$ and $c_n(f)$ be the n-th Fourier coefficient of f by the system Ψ. We denote

$$\mathrm{sp}f = \{n \geq 0 :\ c_n(f) \neq 0\}.$$

Remark 1.1. *If $f \in C[0,1]$ then $c_n(f) \to 0$ and the definition (1.1) is correct, but the condition $f \in L^1(0,1)$ does not imply that coefficients tend to 0, so depending on Γ and f the set $D(f,\Psi,\Gamma)$ can be empty. In this case for $f \in L^1(0,1)$ we split $\mathrm{sp}f$ into two parts*

$$A := \{n \in \mathrm{sp}f :\ |c_n(f)| \leq 1\},\ B := \mathrm{sp}f \setminus A$$

and consider the rearrangements of non-vanishing coefficients for which

$$(1.4)\qquad |c_{\sigma(n)}(f)\gamma_{\sigma(n)}| \searrow \ \text{on } A$$

and the second part can be rearranged arbitrarily. The set $D(f,\Psi,\Gamma)$ and convergence of the approximants (1.2) are considered in this sense.

Note that in the case $\gamma_n \to 0$, the rearrangement satisfying (1.4) always exists. When γ_n does not tend to zero, we consider only functions for which the rearrangement satisfying (1.4) exists. This, as we will see later, does not lose the generality.

Let $\mathrm{H}_p = \{h_{n,p}\}_{n=1}^{\infty}$ be the Haar system normed in $||\cdot||_p$ norm and let $\Gamma_p = \{\gamma_{n,p}\}_{n=1}^{\infty}$, where $\gamma_{n,p} = 2^{-\frac{k}{p}}$, if $n = 2^k + i$, $i = 1,2,...,2^k$, $1 \leq p \leq \infty$. We state a theorem of T. Tao in a slightly different but equivalent form.

Theorem (T. Tao, [22]). *a) If* $1 < p < \infty$ *then for each* $f \in L^p(0,1)$
$$\lim_{N\to\infty} G_N(f, \mathrm{H}_\infty, \Gamma_p) = f(x), \ \text{a.e. on } [0,1].$$
b) There exists $F \in \bigcap_{1<p<\infty} L^p(0,1)$ *such that*
$$\lim_{N\to\infty} \sup |G_N(F, \mathrm{H}_\infty, \Gamma_\infty)(x)| = +\infty, \ x \in [0,1].$$

It is easy to see that $G_N(f, \mathrm{H}_\infty, \Gamma_p) \equiv G_N(f, \mathrm{H}_p)$, $1 \le p \le \infty$, hence this theorem implies that if the Haar functions are normed in $||\cdot||_p$ norm then the approximants (1.2) converges a.e. in the case $1 < p < \infty$ and when $p = \infty$ there exists an integrable function with everywhere divergent greedy algorithm.

For sequence $\Gamma = \{\gamma_n\}_{n=0}^\infty$ we denote

$$\tau(\Gamma) = \sup_{m>n>0} \left\{ \frac{m}{n} : \ \frac{\gamma_n}{\gamma_m} \le 2 \right\}. \tag{1.5}$$

Remark 1.2. *If* $p > 0$ *then* $\tau(\{n^{-p}\}_{n=1}^\infty) < +\infty$, *while* $\tau(\{(\ln n)^{-1}\}_{n=2}^\infty) = +\infty$.

Remark 1.3. *The condition* $\tau(\Gamma) < +\infty$ *implies* $\gamma_n \to 0$. *On the other hand if*
$$\widetilde{\Gamma} = \{\{(\frac{1}{2} + \frac{1}{i})\gamma_{n_k}\}_{i=2}^k\}_{k=2}^\infty, \ \text{where } n_2 < n_3 < ...$$
then clearly $\tau(\widetilde{\Gamma}) = +\infty$. *Besides, if the sequence* n_k, $k = 2, 3, ...$ *tends to* ∞ *sufficiently fast, then* $\widetilde{\Gamma}$ *is monotone and tends to 0 with arbitrary given rate.*

For the Haar system we also have the following

Theorem(S. Gogyan, [10]). *For each* $f \in L^1(0,1)$
$$\lim_{N\to\infty} ||f - G_N(f, \mathrm{H}_1, \Gamma)||_1 = 0$$
if and only if $\tau(\Gamma) < +\infty$.

In the present paper we prove the following theorems.

Theorem 1.4. *The following assertions hold:*

1). *If* $\tau(\Gamma) < +\infty$, *then* $\lim_{N\to\infty} ||f - G_N(f, \mathcal{F}_\infty, \Gamma)||_{\mathrm{C}} = 0$, *for all* $f \in C[0,1]$ *and* $\sigma \in D(f, \mathcal{F}_\infty, \Gamma)$.

2). *If* $\tau(\Gamma) = +\infty$, *then there exists* $f \in C[0,1]$ *such that* $\#D(f, \mathcal{F}_\infty, \Gamma) = 1$ *and* $G_N(f)(x)$ *diverges a.e. on* $[0,1]$.

Theorem 1.5. $\lim_{N\to\infty} G_N(f, \mathcal{F}_\infty, \Gamma)(x) = f(x)$ *a.e. on* $[0,1]$ *for all* $f \in L^1[0,1]$ *and all* $\sigma \in D(f, \mathcal{F}_\infty, \Gamma)$ *if and only if* $\tau(\Gamma) < +\infty$.

Remark 1.6. *Analogues of Theorems* 1.4 *and* 1.5 *for Haar system were proved in* [1]*. By a different approach and in different terms T.W. Körner* [19] *proved an analogue of Theorem* 1.4 *for Haar system when the normalizing coefficients are constant in the blocks of Haar system.*

From Theorems 1.4 *and* 1.5 *we get an analogue of the result of T. Tao for Franklin system.*

2. Definitions and auxiliary results. Uniform convergence

Let

$$\mathcal{F} = \{f_n\}_{n=0}^{\infty} = \{\{f_k^i\}_{i=1}^{2^k}\}_{k=0}^{\infty} \cup \{f_0^0\}, \text{ where } x \in [0,1]$$

be the orthonormal Franklin system (see [12], p.197). It is known (see [12], p. 199) that the Franklin system is a basis in $C[0,1]$ and in $L^p[0,1]$ for $1 \le p < \infty$ and an unconditional basis in $L^p[0,1]$ for $1 < p < \infty$ (see [12], p. 214). Also Fourier-Franklin series of $f \in L^1[0,1]$ converges to f a.e. in $[0,1]$ (see [3] and [4]).

For $n = 2^\mu + \nu$, where $\mu \ge 0$, $1 \le \nu \le 2^\mu$ we denote

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, 0 \le i \le 2\nu, \\ \frac{i-\nu}{2^\mu}, 2\nu < i \le n\ . \end{cases}$$

It is known that f_n is continuous on $[0,1]$ and linear on the intervals $[s_{n,i-1}, s_{n,i}]$, hence it is uniquely defined by values $a_i^{(n)} = f_n(s_{n,i})$. Besides f_n attains its minimum and maximum values on the interval $[s_{n,2\nu-2}, s_{n,2\nu}]$, called a peak interval of f_n and denoted by $\{n\}$ (see [5]).

In the case $2^\mu + 1 < n < 2^{\mu+1}$ the following estimates are true (see [5] and [8]):

$$-\frac{97}{48} a_{2\nu}^{(n)} < a_{2\nu-1}^{(n)} < -\frac{95}{42} a_{2\nu}^{(n)}, \tag{2.1}$$

$$-\frac{107}{66} a_{2\nu-2}^{(n)} < a_{2\nu-1}^{(n)} < -\frac{49}{30} a_{2\nu-2}^{(n)}, \tag{2.2}$$

$$\sqrt{\frac{2}{3}} 2^{\frac{\mu}{2}} \le ||f_n||_\infty = a_{2\nu-1}^{(n)} \le 2^{\frac{\mu}{2}+1}, \tag{2.3}$$

$$\frac{1}{4}|a_{i+1}^{(n)}| \leq |a_i^{(n)}| \leq \frac{2}{7}|a_{i+1}^{(n)}|,\ 1 \leq i \leq 2\nu - 3, \tag{2.4}$$

$$\frac{1}{4}|a_{i-1}^{(n)}| \leq |a_i^{(n)}| \leq \frac{2}{7}|a_{i-1}^{(n)}|,\ 2\nu + 1 \leq i \leq n - 1. \tag{2.5}$$

For $f \in L^1[0,1]$ let $c_n(f) = ||f_n||_\infty^2 \int\limits_0^1 f(x)\widetilde{f}_n(x)dx$ be the n-th coefficient of Fourier-Franklin series of f by the system $\mathcal{F}_\infty$, $n = 0, 1, ...$.

For $n = 2^k + i$, $1 \leq i \leq 2^k$ we denote $[n] = k$ and for an index set $A = \{n_1, ..., n_k\} \subset \mathbb{N}$ denote $[A] = \{[n_1], ..., [n_k]\}$.

The interval of the form $\left(\frac{i-1}{2^k}, \frac{i}{2^k}\right)$ where $i = 1, 2, ..., 2^k$ and $k = 0, 1, ...$ is called a dyadic interval of order k. For $n = 2^k + i$, $i = 1, 2, ..., 2^k$, $k = 0, 1, ...$ we denote

$$\Delta_n = \Delta_k^i = \left(\frac{i-1}{2^k}, \frac{i}{2^k}\right);\ \overline{\Delta}_n = \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right];$$

and

$$\Delta_1 = \Delta_0^0 = (0,1);\ \overline{\Delta}_1 = [0,1].$$

We also denote by $z_n := \frac{2i-1}{2^{k+1}}$ the midpoint of the interval Δ_n, $n = 1, 2, ...$ and by $R^{(2)}$ the set of dyadic rational points from $[0,1]$:

$$R^{(2)} := \bigcup_{k=0}^{\infty} R_k,\ \text{where}\ R_k := \left\{\frac{i}{2^k}\right\}_{i=0}^{2^k},\ k = 0, 1, ...$$

There exist absolute constants $C_0 > 0$ and $0 < r < 1$ so that (see [12], p. 205).

$$|\widetilde{f}_n(x)| \leq C_0 r^{n|x-z_n|},\ x \in [0,1]. \tag{2.6}$$

Using (2.6) we get

$$\sum_{[n]=k} |\widetilde{f}_n(x)| \leq C_1,\ x \in [0,1],\ k = 0, 1, ... \tag{2.7}$$

where C_1 is an absolute constant.

For a polynomial $P(x) = \sum_{n=N}^{M} a_n \widetilde{f}_n(x)$ and a rearrangement σ of the set $\{N, N+1, ..., M\}$ we denote

$$P^*_\sigma(x) = \sup_{N \le k < M} \left| \sum_{n=N}^{k} a_{\sigma(n)} \widetilde{f}_{\sigma(n)}(x) \right|, \ x \in [0,1].$$

In the sequel we denote by C an absolute constant which can be different in different formulas.

The notation $a \asymp b$ means double inequality $C_1 a \le b \le C_2 a$ where $C_1, C_2 > 0$ are absolute constants.

From Lemma 1 in [5] and (2.1) – (2.3) follows

Lemma 2.1. *Let $2^k + 1 < n < 2^{k+1}$. There exists a point $t_n \in \{n\}$, such that*

$$\int_{t_n}^{\frac{\theta}{2^j}} \widetilde{f}_n(t)dt \ge C2^{-k}, \ \textit{if } \{n\} \subset \left[\frac{\theta-1}{2^j}, \frac{\theta}{2^j}\right].$$

Now we use Lemma 2.1 to obtain a result similar to Lemma 2 of [9].

Lemma 2.2. *Let $\Gamma = \{\gamma_n\}_{n=0}^\infty$ be a decreasing sequence of positive numbers and let the dyadic interval*

$$\Delta = \left(\frac{\alpha-1}{2^\beta}, \frac{\alpha}{2^\beta}\right) \subset (0,1),$$

and the index set

$$\Lambda = \{k_i\}_{i=1}^m \subset \mathbb{N}, \ \beta < k_1 < k_2 < ... < k_m, \ m \in \mathbb{N}$$

with

$$\frac{\gamma_A}{\gamma_B} \le 2, \ \textit{where } [2^{k_1}, 2^{k_m+1}] \subset [A, B], \tag{2.8}$$

be given. Suppose $\Omega = \{n \in \mathbb{N} : \ \{n\} \subset \Delta, \ [n] \in \Lambda\} := \{n_i\}_{i=1}^p$, $n_1 < n_2 < ... < n_p$ and let σ be a permutation of Ω satisfying

$$t_{\sigma(n_1)} < t_{\sigma(n_2)} < ... < t_{\sigma(n_p)}, \tag{2.9}$$

where t_n is determined by Lemma 1.

Then the polynomial

$$P(x) := \frac{1}{m} \sum_{i=1}^{p} \frac{\gamma_{\sigma^{-1}(n_i)}}{\gamma_{n_i}} \widetilde{f}_{n_i}(x) \tag{2.10}$$

satisfies the following conditions

1) $\sum_{n=0}^{\infty} |c_n(P) \widetilde{f}_n(x)| \le C, \ x \in [0,1],$

2) $||P||_2 \asymp \sqrt{\frac{|\Delta|}{m}}$,
3) $\sigma \in D(P, \mathcal{F}_\infty, \Gamma)$ *and* $\mu\{x \in \Delta : \ P_\sigma^*(x) > c\} > c|\Delta|$, *for some absolute constant* $c \in (0,1)$.

Proof. Since Γ is decreasing it follows from (2.10) and (2.8) that $\sigma \in D(P, \mathcal{F}_\infty, \Gamma)$ and

$$\frac{1}{2m} \le c_n(P) \le \frac{2}{m}, \ n \in \mathrm{sp}P \tag{2.11}$$

which together with (2.7) and (2.10) proves the first item of the Lemma.

Next, for $[n] = k$ we have $||f_n||_2 = 1$ and $||f_n||_\infty \asymp 2^{\frac{k}{2}}$. Hence $||\widetilde{f}_n||_2 \asymp 2^{-\frac{k}{2}}$ and the assertion 2) of Lemma 2.2 follows from (2.11) and the equality

$$|\{n : \ [n] = k, \{n\} \subset \Delta\}| = 2^{k-\beta}, \ k > \beta.$$

We denote $t_{\sigma(n_{p+1})} = \frac{\alpha}{2^\beta}$. Using (2.11) and Lemma 2.1 we get

$$\int\limits_\Delta P_\sigma^*(x)dx \ge \int\limits_\Delta \sum_{t_{\sigma(n)}<x, \ n\in \mathrm{sp}P} c_{\sigma(n)}(P)\widetilde{f}_{\sigma(n)}(x)dx =$$

$$\sum_{i=1}^{p} \int\limits_{t_{\sigma(n_i)}}^{t_{\sigma(n_{i+1})}} \sum_{t_{\sigma(n)}<x, \ n\in \mathrm{sp}P} c_{\sigma(n)}(P)\widetilde{f}_{\sigma(n)}(x)dx = \sum_{n\in \mathrm{sp}P} \int\limits_{t_n}^{\frac{\alpha}{2^\beta}} c_n(P)\widetilde{f}_n(x)dx \ge$$

$$\frac{1}{3}\frac{C}{m}\sum_{n\in \mathrm{sp}P} 2^{-[n]} = \frac{1}{3}C|\Delta|.$$

From this estimate and the first item of Lemma 2.2 follows that there exists an absolute constant $c \in (0,1)$ such that

$$\mu\{x \in \Delta : \ P_\sigma^*(x) > c\} > c|\Delta|.$$

Lemma 2.2 is proved. □

Remark 2.3. *Let* $\{\tau_n\}_{n=0}^\infty$ *be an arbitrary sequence of numbers with* $\frac{2}{5} \le \tau_n \le \frac{5}{2}$ *and let* Λ *be as in Lemma 2.2. From the proof of Lemma 2.2 follows that the polynomial*

$$P(x) := \frac{1}{m} \sum_{[n]\in\Lambda, \{n\}\subset\Delta} \tau_n \widetilde{f}_n(x) \tag{2.12}$$

satisfies conditions 1) *and* 2). *By Chebishev inequality we have*

$$\mu\{x \in [0,1] : \ |P(x)| > m^{-\frac{1}{4}}\} \le C\frac{|\Delta|}{\sqrt{m}},$$

while using estimates (2.1)−(2.5) *one can see that* $|P(x_0)| > c$, *where* x_0 *is the left endpoint of the interval* Δ *and* $c > 0$ *is an absolute constant.*

Lemma 2.4. *Let* $\Gamma = \{\gamma_n\}_{n=0}^{\infty}$ *be a decreasing sequence of positive numbers with* $\tau(\Gamma) = \infty$. *Given dyadic interval*

$$\Delta = \left(\frac{\alpha - 1}{2^{\beta}}, \frac{\alpha}{2^{\beta}}\right), \quad 1 < \alpha < 2^{\beta}, \quad \beta > 1$$

and numbers $p_0 > \beta$, $M \geq 1$, *there exist an index set* $\Lambda \subset \mathbb{N}$ *of cardinality* $m > M$, *an index set* $J \subset \{n \in \mathbb{N} : \ [n] \in \Lambda, \ \{n\} \subset \Delta\}$ *and a polynomial*

$$Q(x) := \frac{1}{m} \sum_{n \in J} \tau_n \widetilde{f}_n(x), \quad \frac{2}{5} \leq \tau_n \leq \frac{5}{2},$$

such that

1) $\min \Lambda > p_0$,
2) $|Q(x)| \leq C m^{-1/4}, \ x \in [0,1]$,
3) $\# D(Q, \mathcal{F}_{\infty}, \Gamma) = 1$ *and if* $\sigma \in D(Q, \mathcal{F}_{\infty}, \Gamma)$ *then*

$$\mu\{x \in \Delta : \ Q^*_{\sigma}(x) > c\} > c|\Delta|,$$

where $c \in (0,1)$ *is an absolute constant.*

Proof. Observe that for sufficiently large $m > M$ and for any index set Λ of cardinality m with $\min \Lambda > m$, by (2.6) we have

$$\sum_{[n]\in\Lambda, \ \{n\}\subset\Delta} |\widetilde{f}_n(x)| < 1, \ \text{if } \operatorname{dist}(x, \Delta) \geq \frac{1}{2}|\Delta|. \tag{2.13}$$

We fix $\{\tau_n\}_{n=1}^{\infty}$ - an arbitrary sequence of numbers satisfying

$$\frac{2}{5} \leq \tau_n \leq \frac{5}{2}, \quad n = 1, 2, ... \tag{2.14}$$

For sufficiently large $m > M$ with (2.13) and an index $p > m + p_0$ using (2.6) and Remark 2.3 we find p_m with $1 < p_m < m$ such that for

$$\widetilde{Q}(x) := \frac{1}{m} \sum_{k=p+1}^{p+p_m} \sum_{[n]=k, \ \{n\}\subset\Delta} \tau_n \widetilde{f}_n(x) \tag{2.15}$$

we have

$$|\widetilde{Q}(x)| \leq m^{-1/4}, \ x \in [0,1], \tag{2.16}$$

and there exists $x_0 \in [0,1]$ such that

$$\left|\widetilde{Q}(x_0) + \frac{1}{m} \sum_{[n]=p+p_m+1,\ \{n\}\subset\Delta} \tau_n \widetilde{f}_n(x_0)\right| > m^{-1/4}.$$

For $i = 0, 1, ..., m - p_m$ we inductively construct polynomials

$$P_i(x) := \frac{1}{m} \sum_{[n]\in\Lambda_i,\ \{n\}\subset\Delta} \tau_n \widetilde{f}_n(x),$$

$$Q_i(x) := \frac{1}{m} \sum_{n\in J_i} \tau_n \widetilde{f}_n(x),$$

auxiliary sets $E_i^{'}$, such that for the sequence of numbers $\{\varepsilon_i\}_{i=0}^{m-p_m}$, with

$$\varepsilon_0 = m^{-1/4},\ \varepsilon_{i+1} = \varepsilon_i - 2^{-2m}, \tag{2.17}$$

and sets

$$E_i = \{x \in [0,1] :\ |Q_i(x)| > \frac{1}{2}\varepsilon_i\} \tag{2.18}$$

the following properties hold:

a) Λ_i consists of $i + p_m$ different integers with $\Lambda_i > p_0$ and $J_i \subset \{n \in \mathbb{N} :\ [n] \in \Lambda_i, \{n\} \subset \Delta\}$,
b) $\max \Lambda_{i+1} \le \max \Lambda_i + C_1 \log_2 m$,
c) $E_i \subset E_{i+1}$,
d) $E_i \subset E_i^{'} \subset (0,1)$ and $\mu(E_i^{'} \setminus E_i) \le m^{-2}$,
e) $\sum_{n\in \mathrm{sp}Q_{i+1}\setminus \mathrm{sp}Q_i} |\widetilde{f}_n(x)| < \frac{1}{6}2^{-2m}, \quad x \in E_i, \quad i < m - p_m$,
f) $\sum_{n\in \mathrm{sp}P_i\setminus \mathrm{sp}Q_i} |\widetilde{f}_n(x)| < \frac{1}{6}(i+1)2^{-2m}, \quad x \in [0,1] \setminus (E_0^{'} \cup ... \cup E_i^{'})$,
g) $|Q_{i+1}(x)| \le m^{-1/4} + i2^{-2m}, \quad x \in [0,1], \quad i < m - p_m$.

For $i = 0$ we take $\Lambda_0 = \{p+1, p+2, ..., p+p_m\}$, $P_0 = Q_0 = \widetilde{Q}$ where $\widetilde{Q}$ is defined by (2.15) and $J_0 = \mathrm{sp}Q_0$.

Suppose that for $0 \le k < m - p_m$ we have constructed polynomials P_i, Q_i and hence index sets Λ_i, J_i, for $i = 0, 1, ..., k$, also sets $E_i^{'}$ for $i = 0, 1, ..., k-1$ with the properties $a) - g)$. Now we construct the set $E_k^{'}$ and polynomials P_{k+1}, Q_{k+1}.

For the polynomial Q_k we consider the set E_k defined by (2.18). It follows from (2.13) that $dist(E_k, \{0,1\}) \ge \frac{1}{2}|\Delta|$. Since Q_k is a continuous piecewise linear function, E_k is an open set in $(0,1)$. We represent

E_k as a union of intervals

$$E_k = \bigcup_{j=1}^{N_k} (a_k^j, b_k^j) := \bigcup_{j=1}^{N_k} \Delta_k^{j,0}$$

and denote $\max \Lambda_k := u_k$. From the definition of Q_k and Franklin functions we have $N_k \le 2^{u_k+2}$. Now, for $j = 1, 2, ..., N_k$ we enlarge each interval $\Delta_k^{j,0}$ to the interval $\Delta_k^{j,1} := (c_k^j, d_k^j)$ such that

$$0 < a_k^j - \frac{m^{-2}}{4N_k} \le c_k^j \le a_k^j - \frac{m^{-2}}{8N_k}, \tag{2.19}$$

$$b_k^j + \frac{m^{-2}}{8N_k} \le d_k^j \le b_k^j + \frac{m^{-2}}{4N_k} < 1. \tag{2.20}$$

We also choose endpoints of $\Delta_k^{j,1}$ to be from $R_{k'}$ for some $k' > u_k$. Since $N_k \le 2^{u_k+2}$, by $(2.19) - (2.20)$ we can take

$$k' < u_k + C_1 \log_2 m. \tag{2.21}$$

Let G_k be the union of enlarged intervals. From $(2.19) - (2.20)$ follows

$$E_k \subset G_k,\ \overline{G_k} \subset (0,1) \text{ and } \mu(G_k \setminus E_k) \le \frac{1}{2} m^{-2}. \tag{2.22}$$

Now we enlarge each interval $\Delta_k^{j,1}$ by the same way as we did for $\Delta_k^{j,0}$ and denote the obtained intervals by $\Delta_k^{j,2}$, $j = 1, 2, ..., N_k$. We also take the endpoints of each $\Delta_k^{j,2}$ from $R_{k''}$ for some $k'' > k'$, where $k'' < u_k + C_1 \log_2 m$. Now we take $E_k' = \bigcup_{j=1}^{N_k} \Delta_k^{j,2}$ and observe that after the enlargement we have

$$G_k \subset E_k',\ \overline{E_k'} \subset (0,1) \text{ and } \mu(E_k' \setminus G_k) \le \frac{1}{2} m^{-2}. \tag{2.23}$$

It follows from the construction of G_k and E_k' that

$$\min(dist(\partial G_k, \partial E_k'),\ dist(\partial E_k', \partial E_k)) \ge \frac{m^{-2}}{8N_k} \ge \frac{m^{-2}}{2^{u_k+5}}. \tag{2.24}$$

We take an index $k_1 > k''$ and denote

$$g_k(x) := \frac{1}{m} \sum_{[n]=k_1, \{n\} \subset \Delta \setminus G_k} \tau_n \widetilde{f}_n(x),$$

$$g_k'(x) := \frac{1}{m} \sum_{[n]=k_1, \{n\} \subset \Delta \cap G_k} \tau_n \widetilde{f}_n(x).$$

By (2.6) and (2.24) we can choose $k_1 < u_k + C_1 \log_2 m$ so that

$$\frac{1}{m} \sum_{n \in \mathrm{sp} g_k} |\widetilde{f}_n(x)| < \frac{1}{6} 2^{-2m}, \ x \in E_k, \tag{2.25}$$

$$\frac{1}{m} \sum_{n \in \mathrm{sp} g_k^{'}} |\widetilde{f}_n(x)| < \frac{1}{6} 2^{-2m}, \ x \in [0,1] \setminus E_k^{'}. \tag{2.26}$$

Now we take $\Lambda_{k+1} := \Lambda_k \cup \{k_1\}$, $P_{k+1} := P_k + g_k + g_k^{'}$ and $Q_{k+1} := Q_k + g_k$ and prove that conditions $a) - g)$ for $i = k$ are fulfilled.

By the induction hypothesis and definition of Λ_{k+1} we obtain a). From the choice of integers $k^{'}$, $k^{''}$ and k_1 we get b). The item c) follows from (2.18), definition of Q_{k+1}, (2.14) and (2.25). Next, from (2.22) and (2.23) we get d). Then, since $\mathrm{sp}Q_{k+1} = \mathrm{sp}Q_k \cup \mathrm{sp}g_k$ we get $e)$ from (2.25). Now observe that by the construction of P_{k+1} and Q_{k+1} we have

$$\mathrm{sp}P_{k+1} \setminus \mathrm{sp}Q_{k+1} = (\mathrm{sp}P_k \setminus \mathrm{sp}Q_k) \cup \mathrm{sp}g_k^{'},$$

hence by the induction hypothesis and (2.26) we obtain $f)$. Finally, by the induction hypothesis, (2.18), (2.25) and (2.7) we obtain $g)$.

Now applying the process described above for $i = 0$ we get the validity of the statement of induction in the case $i = 0$.

On the last step when $i = m - p_m$ we denote $N := m - p_m$, $P := P_N$, $Q := Q_N$ and $\Lambda := \Lambda_N$ and $J := J_N$.

By f) and (2.14) for each $x \in [0,1] \setminus (E_0^{'} \cup E_1^{'} \cup ... \cup E_N^{'})$ we have

$$\sum_{n \in \mathrm{sp}P \setminus \mathrm{sp}Q} |c_n(P) \widetilde{f}_n(x)| \le 2^{-m}. \tag{2.27}$$

On the other hand from c) and d) we have

$$\mu((E_0^{'} \cup ... \cup E_N^{'}) \vartriangle E_N) \le \mu((E_0^{'} \cup ... \cup E_{N-1}^{'}) \setminus E_N) + m^{-2} \le$$
$$m^{-2} + \mu((E_0^{'} \cup ... \cup E_{N-1}^{'}) \vartriangle E_{N-1}) \le ... \le m^{-1}$$

hence

$$\mu(E_0^{'} \cup ... \cup E_N^{'}) \le \mu(E_N) + \frac{1}{m}. \tag{2.28}$$

From b) follows that $[\mathrm{sp}P] \subset \overline{p, p+m^2}$, if $m > M$ is sufficiently large and $p > m + p_0$. Since $\tau(\Gamma) = \infty$, there exist integers $B > A > 2^{m+p_0}$ such that

$$\frac{B}{A} > 2^{m^2} \text{ and } \frac{\gamma_A}{\gamma_B} \le 2.$$

Hence if we start the construction of Λ from $p = [\log_2 A] + 1$, we will have sp$P \subset \overline{A, B}$ for any sequence $\{\tau_n\}_{n=1}^{\infty}$ with (2.14).

Now suppose $\overline{A, B} = \{n_i\}_{i=1}^{q}$, where $n_1 < n_2 < ... < n_q$. We take

$$\tau_{n_i} = \frac{\gamma_{\sigma^{-1}(n_i)}}{\gamma_{n_i}}, \ i = 1, 2, ..., q \tag{2.29}$$

where σ is a permutation of the set $\overline{A, B}$ such that

$$t_{\sigma(n_1)} < t_{\sigma(n_2)} < ... < t_{\sigma(n_q)},$$

and t_n is determined by Lemma 2.1. From the choice of the numbers A and B, monotonicity of the sequence Γ and (2.29) we have $\frac{1}{2} \leq \tau_n \leq 2$, $n \in \overline{A, B}$ and $\sigma \in D(P, \mathcal{F}_\infty, \Gamma) \cap D(Q, \mathcal{F}_\infty, \Gamma)$. As we see, P is the polynomial from Lemma 2.2 for the index set Λ, hence P satisfies items $2 - 3$ of Lemma 2.2.

Since $p > p_0$ the first item of Lemma 2.4 is fulfilled. From g) we get the second item of Lemma 2.4. From the second item of Lemma 2.2, (2.28) and the Chebishev inequality we have

$$\mu(E_0' \cup ... \cup E_N') \leq Cm^{1/2}||Q||_2^2 + \frac{1}{m} \leq Cm^{1/2}||P||_2^2 + \frac{1}{m} \leq C\frac{|\Delta|}{\sqrt{m}} + \frac{1}{m}. \tag{2.30}$$

Since P satisfies the third item of Lemma 2.2, from (2.27) and (2.30) we get

$$\mu\{x \in (0, 1) : \ Q_\sigma^*(x) > c\} > c|\Delta|.$$

To complete the proof observe that if needed we can slightly perturb the coefficients of P so that all the quantities $|c_n(P)\gamma_n|$ will be different from each other and $\frac{2}{5} \leq \tau_n \leq \frac{5}{2}$, for $n \in$ sp$P$ by the same time all mentioned properties of polynomials P, Q and the permutation σ will be preserved. Hence we can suppose that $\#D(Q, \mathcal{F}_\infty, \Gamma) = 1$.

Lemma 2.4 is proved. □

3. Auxiliary results. Pointwise convergence

Theorem (G. Gevorkyan, [6]). *The series with respect to Franklin system*

$$\sum_{n=0}^{\infty} a_n f_n(x)$$

converges a.e. on a set $E \subset (0,1)$ *of positive measure if and only if* $\sum_{n=0}^{\infty} a_n^2 f_n^2(x) < +\infty$ *a.e. on* E.

Lemma 3.1. *Let* $E \subset [0,1]$ *with* $\mu(E) > 0$ *and let* $0 \le n_1 < n_2 < ...$ *be a sequence of nonnegative integers such that* $\sum_{i=0}^{\infty} \widetilde{f}^2_{n_i}(x) < +\infty$ *a.e. on* E. *Then*

$$\sum_{i=0}^{\infty} |\widetilde{f}_{n_i}(x)| < +\infty \text{ a.e. on } E.$$

Proof. We use the technique from [6]. Let $\delta > 0$. There exists a compact set $A \subset E$ such that

$$\sum_{i=0}^{\infty} \widetilde{f}^2_{n_i}(x) \le M < +\infty, \; x \in A \tag{3.1}$$

and

$$\mu(A) > \mu(E) - \delta. \tag{3.2}$$

Let $\{I_k^j\}$, where $k = 1, 2, ...$ and $j \in \{1, 2, ..., 2^k\}$, be a family of dyadic intervals which are defined as follows. For $k = 1$ we split $[0,1]$ into intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 1)$ and denote by I_1^j those intervals for which

$$\frac{\mu(I_1^j \cap A)}{\mu(I_1^j)} < \delta. \tag{3.3}$$

Suppose we have chosen intervals I_k^j for $k = 1, 2, ..., N$. We split $[0,1]$ into intervals $(\frac{j-1}{2^{N+1}}, \frac{j}{2^{N+1}})$, $j = 1, 2, ..., 2^{N+1}$ and denote by I_{N+1}^j those intervals which do not intersect with any I_k^j for $k \le N$ and

$$\frac{\mu(I_{N+1}^j \cap A)}{\mu(I_{N+1}^j)} < \delta. \tag{3.4}$$

Let $B = \bigcup_{k=1}^{\infty} \bigcup_j I_k^j$. Clearly $[0,1] \setminus B \subset A$ with the exception of at most dyadic rational points, also

$$\mu(E \setminus ([0,1] \setminus B)) < 2\delta. \tag{3.5}$$

Now we denote by $\{n_j^{'} : j = 0, 1, ...\}$ the subset of the numbers $\{n_i : i = 0, 1, ...\}$ with $\{n_j^{'}\} \subset B$ and the rest we denote by $\{n_j^{''} : j = 0, 1, ...\}$.

For $x \in I_m^j$ and $k \ge m$ from (2.6) we have

$$\sum_{[n_j^{''}]=k} \widetilde{f}^2_{n_j^{''}}(x) \le C \sum_{[n_j^{''}]=k} r^{n_j^{''} |x - z_{n_j^{''}}|}. \tag{3.6}$$

Since $z_{n_j''} \notin I_m^j$ and $z_{n_j''}$ takes values of the form $\frac{p}{2^k}$, from (3.6) we get

$$\sum_{[n_j'']=k} \widetilde{f}^2_{n_j''}(x) \le C \sum_{q \ge 2^k \rho(x, I_{k_0}^{j,c})} r^q \le C r^{2^k \rho(x, I_{k_0}^{j,c})}, \text{ for } x \in I_{k_0}^j, \tag{3.7}$$

where $\rho(x, I_{k_0}^{j,c})$ is the distance of x from the complement of $I_{k_0}^j$.

From (3.7) follows

$$\int_{I_m^j} \sum_{k \ge m} \sum_{[n_j'']=k} \widetilde{f}^2_{n_j''}(x)dx \le C \sum_{k \ge m} \int_{I_m^j} r^{2^k \rho(x, I_m^{j,c})} dx \le C \sum_{k \ge m} \int_0^1 r^{2^k t} dt \le$$

$$C \sum_{k \ge m} \frac{1}{2^k} \int_0^\infty r^t dt \le C\mu(I_m^j). \tag{3.8}$$

Now we show that

$$\int_0^1 \sum_{n_j''} \widetilde{f}^2_{n_j''}(x)dx < +\infty. \tag{3.9}$$

From (3.1), (3.8) and (2.7) we obtain

$$\int_0^1 \sum_{n_j''} \widetilde{f}^2_{n_j''}(x)dx = \int_{[0,1]\setminus B} \sum_{n_j''} \widetilde{f}^2_{n_j''}(x)dx + \int_B \sum_{n_j''} \widetilde{f}^2_{n_j''}(x)dx \le$$

$$M + \sum_{m=0}^{\infty} \sum_i \int_{I_m^i} \sum_{k=0}^{\infty} \sum_{[n_j'']=k} \widetilde{f}^2_{n_j''}(x)dx \le M' + C \sum_{m=0}^{\infty} \sum_i \mu(I_m^i) +$$

$$\sum_{m=0}^{\infty} \sum_i \int_{I_m^i} \sum_{k=0}^{m-2} \sum_{[n_j'']=k} \widetilde{f}^2_{n_j''}(x)dx. \tag{3.10}$$

Let J be a dyadic interval of order $m-1$ which contains I_m^i. It follows from the definition of the intervals I_k^j that

$$\frac{\mu(J \cap A)}{\mu(J)} \ge \delta. \tag{3.11}$$

Denote $F(x) = \sum_{k=0}^{m-2} \sum_{[n_j'']=k} \widetilde{f}^2_{n_j''}(x)$. From the definition of the functions of the Franklin system we have that $F(x)$ is a quadratic polynomial on the interval J. Now for $k = [\log_2 \frac{8}{\delta}] + 1$ we split J into dyadic intervals $\{J_i\}$ of the length $\frac{|J|}{2^k}$. Using (3.11) we get

$$\#\{i : \ \mu(A \cap J_i) \geq \frac{\delta}{2}|J_i|\} \geq 4 \tag{3.12}$$

and by the definition of the intervals J_i we have

$$\frac{\delta}{16}|J| \leq |J_i| \leq \frac{\delta}{8}|J|. \tag{3.13}$$

From (3.12) and (3.13) follows that there exist points $\{a_i\}_{i=1}^4 \subset A \cap J$ with

$$a_{i+1} - a_i \geq \frac{\delta^2}{32}|J|, \ i = 1, 2, 3, \tag{3.14}$$

which together with (3.1) implies

$$|F'(c_i)| \leq \frac{C}{\delta^2}\frac{1}{|J|}, \ i = 1, 2, \tag{3.15}$$

where

$$c_1, c_2 \in J \text{ and } |c_1 - c_2| \geq \frac{\delta^2}{32}|J|. \tag{3.16}$$

Since $F'(x)$ is linear on the interval J, from (3.15) and (3.16) we obtain

$$|F'(x)| \leq \frac{C}{\delta^4}\frac{1}{|J|}, \ x \in J. \tag{3.17}$$

From (3.1), (3.11) and (3.17) we get

$$|F(x)| \leq \frac{C}{\delta^4}, \ x \in J, \tag{3.18}$$

which together with (3.10) implies (3.9).

Observe that by (2.6) we have $\sum_{k=1}^{\infty}(|\widetilde{f}_k^1(x)| + |\widetilde{f}_k^{2^k}(x)|) < +\infty$ a.e. on $[0, 1]$ hence we can suppose that $2^{[n_i]} + 1 < n_i < 2^{[n_i]+1}$, $i = 0, 1, \ldots.$ Each $\widetilde{f}_n$ is piecewise linear on $[0, 1]$, hence by (2.1) – (2.5) we have

(3.19)
$$\int_0^1 |\widetilde{f}_{n''_j}(x)|dx \leq C \int_{\{n''_j\}} |\widetilde{f}_{n''_j}(x)|dx \leq C' \int_{\{n''_j\}} |\widetilde{f}_{n''_j}(x)|^2 dx \leq C' \int_0^1 |\widetilde{f}_{n''_j}(x)|^2 dx.$$

From (3.9), (3.19) and Levi's theorem we get

$$\sum_{n''_j} |\widetilde{f}_{n''_j}(x)| < +\infty \text{ a.e. on } [0,1]. \tag{3.20}$$

Let γ be a positive number. We denote by $\widetilde{I}^i_k$ the concentric interval with I^i_k with $\mu(\widetilde{I}^i_k) = (1+2\gamma)\mu(I^i_k)$ and let $\widetilde{B} = \bigcup_{k=0}^{\infty}\bigcup_i \widetilde{I}^i_k$. Clearly

$$\mu(\widetilde{B}) \le (1+2\gamma)\mu(B) \tag{3.21}$$

Next, for fixed m and $k \ge m$ we have

$$\#\{j:\ [n'_j] = k,\ \{n'_j\} \subset I^i_m\} \le 2^k \mu(I^i_m), \tag{3.22}$$

hence from (2.6) and (3.22) we have

$$\sum_{[n'_j]=k,\ \{n'_j\}\subset I^i_m} |\widetilde{f}_{n'_j}(x)| \le C2^k \mu(I^i_m) r^{2^k \rho(x, I^i_m)},\ x \notin I^i_m, \tag{3.23}$$

where $\rho(x, I^i_m)$ is the distance of x from the interval I^i_m. Using (3.23) we get

$$\int\limits_{[0,1]\setminus\widetilde{B}} \sum_{n'_j} |\widetilde{f}_{n'_j}(x)| dx = \int\limits_{[0,1]\setminus\widetilde{B}} \sum_{m=0}^{\infty}\sum_i \sum_{k=0}^{\infty} \sum_{[n'_j]=k,\ \{n'_j\}\subset I^i_m} |\widetilde{f}_{n'_j}(x)| dx \le$$

$$\sum_{m=0}^{\infty}\sum_i \sum_{k=m}^{\infty} \int\limits_{[0,1]\setminus\widetilde{I}^i_m} \sum_{[n'_j]=k,\ \{n'_j\}\subset I^i_m} |\widetilde{f}_{n'_j}(x)| dx \le$$

$$C\sum_{m=0}^{\infty}\sum_i \mu(I^i_m) \sum_{k=m}^{\infty} 2^k \int\limits_{\gamma|I^i_m|}^{\infty} r^{2^k t} dt < +\infty. \tag{3.24}$$

From (3.24) and Levi's theorem we get

$$\sum_{n'_j} |\widetilde{f}_{n'_j}(x)| < +\infty \text{ a.e. on } [0,1]\setminus\widetilde{B}.$$

Since γ is arbitrary positive number we get

$$\sum_{n'_j} |\widetilde{f}_{n'_j}(x)| < +\infty \text{ a.e. on } [0,1]\setminus B. \tag{3.25}$$

Taking into account that δ is arbitrary from (3.25), (3.2) and (3.5) we get

$$\sum_{n'_j} |\widetilde{f}_{n'_j}(x)| < +\infty \text{ a.e. on } E, \tag{3.26}$$

which combined with (3.20) implies

$$\sum_{i=0}^{\infty} |\widetilde{f}_{n_i}(x)| < +\infty \text{ a.e. on } [0,1].$$

Lemma 3.1 is proved. □

4. Proofs of the theorems

Proof of Theorem 1. First we prove assertion 1). Let $\tau(\Gamma) < \infty$, $f \in C[0,1]$ and let the number $\varepsilon > 0$ be fixed. Denote

$$T_\varepsilon(f)(x) := \sum_{|c_n(f)\gamma_n| > \varepsilon} c_n(f)\widetilde{f}_n(x), \; x \in [0,1], \tag{4.1}$$

and

$$N(\varepsilon) := \min\{N \in \mathbb{N} : \; |c_n(f)\gamma_n| \le \varepsilon, \; \forall n > N\}.$$

Then

$$\{n \in \mathbb{N} : \; |c_n(f)\gamma_n| > \varepsilon\} \subset \{1, 2, ..., N(\varepsilon)\}, \tag{4.2}$$

$$\frac{\varepsilon}{\gamma_{N(\varepsilon)}} \le |c_{N(\varepsilon)}| \to 0, \text{ as } \varepsilon \to 0. \tag{4.3}$$

Using (2.7) and the monotonicity of Γ we get

$$||S_{N(\varepsilon)}(f) - T_\varepsilon(f)||_C = \left\| \sum_{n \le N(\varepsilon), \; |c_n(f)\gamma_n| \le \varepsilon} c_n(f)\widetilde{f}_n(x) \right\|_C \le$$

$$C\varepsilon \sum_{k=0}^{[\log_2 N(\varepsilon)]-1} \frac{1}{\gamma_{n_k}} + C\frac{\varepsilon}{\gamma_{N(\varepsilon)}}, \tag{4.4}$$

where $n_k = 2^{k+1}$. If $l_0 := [\log_2 \tau(\Gamma)]+1$, then

$$\frac{n_{k+l_0}}{n_k} > \tau(\Gamma), \; k = 0, 1, ..., [\log_2 N(\varepsilon)] - l_0 - 1,$$

which implies

$$\frac{\gamma_{n_k}}{\gamma_{n_{k+l_0}}} > 2, \ k = 0, 1, ..., [\log_2 N(\varepsilon)] - l_0 - 1, \tag{4.5}$$

therefore

$$\sum_{k=0}^{[\log_2 N(\varepsilon)]-1} \frac{1}{\gamma_{n_k}} \leq \sum_{r=0}^{l_0-1} \sum_{k \equiv r (mod \ l_0)} \frac{1}{\gamma_{n_k}} \leq$$

$$\sum_{k=0}^{l_0-1} \left(\frac{1}{2^{i_r}} + \frac{1}{2^{i_r-1}} + ... + 1 \right) \frac{1}{\gamma_{N(\varepsilon)}} \leq \frac{C}{\gamma_{N(\varepsilon)}}, \tag{4.6}$$

for some indices i_r, $r = 0, 1, ..., l_0 - 1$.

From (4.4), (4.6) and (4.3) follows that

$$||S_{N(\varepsilon)}(f) - T_\varepsilon(f)||_C \to 0, \ \text{as } \varepsilon \to 0.$$

Since $||f - S_N(f)||_C \to 0$, we get

$$||f - T_\varepsilon(f)||_C \to 0, \ \text{as } \varepsilon \to 0. \tag{4.7}$$

If the set $D(f, \mathcal{F}_\infty, \Gamma)$ contains only one permutation, then for any $N \in \mathbb{N}$ there exists $\varepsilon = \varepsilon(N) > 0$ such that $G_N(f) \equiv T_\varepsilon(f)$, hence $\lim\limits_{N\to\infty} ||f - G_N(f)||_C = 0$. In the case $\# D(f, \mathcal{F}_\infty, \Gamma) > 1$ for $n = 0, 1, ...,$ we denote

(4.8)

$$\Omega_{-1} = \emptyset, \ \Omega_n(f) := \Omega_n = \{k \in \mathbb{Z}_+ \backslash (\Omega_0 \cup ... \cup \Omega_{n-1}) : \ |\gamma_k c_k(f)| = |\gamma_n c_n(f)|\},$$

and if $\Omega_n \neq \emptyset$, we denote $\omega_n = \max \Omega_n$. Now if $\# \Omega_n > 1$ using the same ideas as we show (4.6) we will have

$$\sum_{k \in \Omega_n} |c_k(f) \widetilde{f}_k(x)| = |\gamma_{\omega_n} c_{\omega_n}(f)| \sum_{k \in \Omega_n} \frac{1}{\gamma_k} |\widetilde{f}_k(x)| \leq$$

$$|\gamma_{\omega_n} c_{\omega_n}(f)| \sum_{k=0}^{\omega_n} \frac{1}{\gamma_k} |\widetilde{f}_k(x)| \leq C |c_{\omega_n}(f)|, \ x \in [0, 1], \tag{4.9}$$

which together with (4.7) completes the proof of the first part of Theorem 1.4.

Now we proceed to prove assertion 2).

Let $\tau(\Gamma) = \infty$. We denote $\{\Delta_{n_j}\}_{j=1}^\infty := \{\{\Delta_k^i\}_{i=2}^{2^k-1}\}_{k=2}^\infty$ and $M_j = 2^{-8j}$, $j = 1, 2, ...$ Let Q_1 be the resulting polynomial of Lemma 2.4 with initial conditions $\Delta = \Delta_{n_1}$, $p_0 = 1$ and $M = M_1$. We put $p_1 = 1$ and for $j = 2, 3, ...$ inductively define polynomials Q_j and numbers p_j where

$$p_j = p_{j-1} + \max \mathrm{sp} Q_{j-1} + 1 \tag{4.10}$$

and Q_j is the resulting polynomial of Lemma 2.4 with initial conditions $\Delta = \Delta_{n_j}$, $p_0 = p_j$ and $M = M_j$.

Next, for $k = 1, 2, ...$ we denote

$$f_k := \sum_{|\Delta_{n_j}|=2^{-k}} Q_j. \tag{4.11}$$

From the choice of the numbers M_j and the second condition of Lemma 2.4 follows

$$||f_k||_{\mathrm{C}} \leq C2^{-k}. \tag{4.12}$$

Since $\#D(Q_j, \mathcal{F}_\infty, \Gamma) = 1$ for $j = 1, 2, ...$, from (4.10) and monotonicity of Γ we have $\#D(f_k, \mathcal{F}_\infty, \Gamma) = 1$, for $k = 1, 2, ...$

We denote

$$f(x) := \sum_{k=1}^{\infty} f_k(x),\ x \in [0, 1]$$

and prove that f satisfies the theorem.

First of all from (4.12) follows that $f \in C[0, 1]$. From construction of f follows that $\#D(f, \mathcal{F}_\infty, \Gamma) = 1$ and if $\sigma \in D(f, \mathcal{F}_\infty, \Gamma)$ then

$$\sigma|_{\mathrm{sp}Q_j} \in D(Q_j, \mathcal{F}_\infty, \Gamma), \quad j = 1, 2, ... \tag{4.13}$$

Suppose that for some $E \subset (0, 1)$ with $\mu(E) > 0$ the limit

$$\lim_{N\to\infty} G_N(f)(x) \text{ exists if } x \in E.$$

According to the Egorov theorem there exists a set $E_0 \subset E$ of positive measure such that $G_N(f)(x)$ converges uniformly on E_0. Take a density point $x_0 \in E_0$ and denote $B = \{j \in \mathbb{N} :\ x_0 \in \Delta_{n_j}\}$. From the definition of density point, uniform convergence of the greedy approximants on E_0 and (4.13) follows that there exists a set $B_1 \subset B$ such that if $j \in B_1$ then

$$\mu(\Delta_{n_j} \cap E_0) > (1 - c)\mu(\Delta_{n_j}), \tag{4.14}$$

and

$$|(Q_j)^*_\sigma(x)| < c,\ x \in E_0, \tag{4.15}$$

where c is a constant from the third item of Lemma 2.4.

The inequalities (4.14) and (4.15) contradicts the third item of Lemma 2.4.

Theorem 1 is proved. □

Proof of Theorem 2. The necessity part obviously follows from the first part of Theorem 1. Now we prove the sufficiency. Since the condition $f \in L^1(0,1)$ does not imply that Fourier-Franklin coefficients of f tend to 0, the methods from the proof of Theorem 1 do not work in this case. In order to apply the technique of Theorem 1 we adjust the set of coefficients of f in appropriate way.

Let $\tau(\Gamma) < \infty$ and $f \in L^1(0,1)$. We denote

$$A_0 = \emptyset,\ A_n = \{k \in \mathbb{Z}_+ \setminus (A_1 \cup ... \cup A_{n-1}) :\ |c_k(f)| \geq \frac{1}{n}\},\ n = 1, 2, ...$$

We will consider only A_n with infinite cardinality. For this aim denote

$$\psi(n) = \begin{cases} 0, & \text{if} \quad \#A_n < \infty, \\ n, & \text{if} \quad \#A_n = \infty\ . \end{cases}$$

From the definition of $\psi(n)$ follows that $A_{\psi(n)}$ is either empty or an infinite set. Since the Fourier-Franklin series of f converges to f a.e. on $[0,1]$, by Theorem A we get

$$\sum_{k \in A_{\psi(n)}} \widetilde{f}_k^2(x) \leq n^2 \sum_{k \in A_{\psi(n)}} |c_k(f) \widetilde{f}_k(x)|^2 < +\infty \text{ a.e. on } [0,1],$$

which together with Lemma 3.1 implies that convergence in the blocks $A_{\psi(n)}$ is absolute.

Now we fix an arbitrary $\delta > 0$ and let $N_0 > 1$ be chosen so that $|c_k(f)| < \delta$ for any $k \in A_{\psi(n)}$ with $n \geq N_0$. Denote $B := \mathbb{Z}_+ \setminus (A_{\psi(1)} \cup ... \cup A_{\psi(N_0)})$ and

$$N(\varepsilon) := \min\{N \in B :\ |c_n(f)\gamma_n| \leq \varepsilon,\ \forall n > N,\ n \in B\},\ \varepsilon > 0.$$

Clearly

$$\{n \in B :\ |c_n(f)\gamma_n| > \varepsilon\} \subset \{1, 2, ..., N(\varepsilon)\}, \tag{4.16}$$

$$\frac{\varepsilon}{\gamma_{N(\varepsilon)}} \leq |c_{N(\varepsilon)}(f)|. \tag{4.17}$$

Now we split the sums $T_\varepsilon(f)(x)$ and $S_{N(\varepsilon)}(f)(x)$, where $T_\varepsilon(f)$ is defined as (4.1), into two parts in the following way:

$$T_\varepsilon(f)(x) = \sum_{n\in B:|c_n(f)\gamma_n|>\varepsilon} c_n(f)\widetilde{f}_n(x) + \Sigma_1,$$

$$S_{N(\varepsilon)}(f)(x) = \sum_{n=0,\ n\in B}^{N(\varepsilon)} c_n(f)\widetilde{f}_n(x) + \Sigma_2.$$

We have

$$|T_\varepsilon(f)(x) - S_{N(\varepsilon)}(f)(x)| \le |\Sigma_1 - \Sigma_2| +$$

$$\left| \sum_{n\in B:\ |c_n(f)\gamma_n|>\varepsilon} c_n(f)\widetilde{f}_n(x) - \sum_{n=0,\ n\in B}^{N(\varepsilon)} c_n(f)\widetilde{f}_n(x) \right|.$$

Since N_0 is fixed and the convergence in the blocks $A_{\psi(n)}$ is absolute, we get that the first difference tends to 0 a.e. on $[0,1]$ as $\varepsilon \to 0$. By the same methods as in the proof of the first part of Theorem 1 we get that the second difference is less than $C|c_{N(\varepsilon)}(f)| \le C\delta$, when $\varepsilon > 0$ is sufficiently small. Since δ is arbitrary we get that $T_\varepsilon(f)(x)$ converges to f a.e. on $[0,1]$ as $\varepsilon \to 0$.

As in the proof of the first part of Theorem 1.4 observe that if the set $D(f, \mathcal{F}_\infty, \Gamma)$ contains only one permutation, then for any $N \in \mathbb{N}$ there exists $\varepsilon = \varepsilon(N) > 0$ such that $G_N(f) \equiv T_\varepsilon(f)$, hence $\lim\limits_{N\to\infty} G_N(f)(x) = f(x)$ a.e. on $[0,1]$. In the case $\#D(f, \mathcal{F}_\infty, \Gamma) > 1$ fix $m \in \mathbb{N}$ and denote $B_m = A_{\psi(1)} \cup ... \cup A_{\psi(m)}$, $g_m = f - \sum\limits_{n\in B_m} c_n(f)\widetilde{f}_n$ and let $\Omega_n(g_m)$ be defined as in (4.8), $n = 0, 1, 2, ...$. Since the convergence in the blocks $A_{\psi(n)}$ is absolute the proof completes once we observe that

$$\lim_{m\to\infty} \lim_{n\to\infty} \sup \sum_{k\in\Omega_n(g_m)} |c_k(g_m)\widetilde{f}_k(x)| = 0 \text{ a.e. on } [0,1].$$

The latter follows from (4.9) and the definition of the sets B_m and $A_{\psi(n)}$.

Theorem 2 is proved. $\square$

Remark 4.1. *The methods developed in the paper are applicable for wavelet systems. For wavelet systems with the order of decay* $\frac{C}{(1+|x|)^{2+\varepsilon}}$ *for some* $\varepsilon > 0$*, in* [9] *a function from* $L^\infty(\mathbb{R})$ *was constructed such that greedy algorithm in* L^∞ *diverges a.e. Putting together the approach developed in Lemma 2 of* [9] *and our Lemmas* 2.2 *and* 2.4 *we can construct a continuous function with the same property, if the wavelet in addition is continuous.*

Acknoledgment. The author would like to express his gratitude to Professor Arthur Sahakyan for his help in the preparation of this paper and valuable remarks which eventually lead to a simplification of many arguments presented here.

References

[1] H. Aleksanyan, "On the greedy algorithm by the Haar system", Journal of Contemporary Mathematical Analysis, **45** (3), 151-161 (2010)

[2] G. Amirkhanyan, "Convergence of greedy algorithm in Walsh system in L_p", Journal of Contemporary Mathematical Analysis, **43** (3), 127-134 (2008)

[3] Z. Chiesielski, "Properties of the orthonormal Franklin system", Studia Math., **23**, 141-157, (1963)

[4] Z. Chiesielski, "Properties of the orthonormal Franklin system. II", Studia Math., **27**, 289-323, (1966)

[5] G.G. Gevorkian, "On absolute and unconditional convergence of Franklin series", Mat. Zamteki (in Russian), **45** (3), 30-42 (1989)

[6] G.G. Gevorkian, "On series with respect to the Franklin system", Analysis Math. (in Russian), **16** (2), 86-114 (1990)

[7] G.G. Gevorkian, "Unboundedness of shift operator with Franklin system in space L_1", Mat. Zametki (in Russian), **38** (4), 523-533 (1985)

[8] G.G. Gevorkian, "Weyl multipliers for the unconditional convergence of series in a Franklin system", Mat. Zametki (in Russian), **41** (6), 789-797 (1985)

[9] G.G. Gevorkian, A.A. Stepanyan, "Almost everywhere divergence of a summation method for wavelet expansions", Journal of Contemporary Mathematical Analysis **39** (2), 33-39 (2004)

[10] S. Gogyan, "On Greedy Algorithm in $L^1(0,1)$ with regard to subsystems of Haar system and on ω-quasi-greedy bases", Math. Notes, **88** (1), 16-27, 2010

[11] M. Grigoryan, A. Sargsyan, "Nonlinear approximation of continuous functions with regard to Faber-Schauder system", Mat. Sbornik (in Russian), **199** (5), 3-26 (2008)

[12] B.S. Kashin, A.A. Saakyan, "**Orthogonal series**", AMS, 1989

[13] G. Kerkacharian, D. Picard, V.N. Temlyakov, "Some inequalities for the tensor product of greedy bases and weight-greedy bases", East J. on Approx., **12** (1), 103-118 (2006)

[14] S.V. Konyagin, V.N. Temlyakov, "A remark on greedy approximation in Banach spaces", East J. on Approx. **5** (3), 365-379 (1999)

[15] S.V. Konyagin, V.N. Temlyakov, "Convergence of greedy approximation I. General Systems", Studia Math. **159** (1), 143-160 (2003)

[16] S.V. Konyagin, V.N. Temlyakov, "Convergence of greedy approximation II. The Trigonomeric System", Studia Math. **159** (2), 161-184 (2003)

[17] T.W. Körner, "Divergence of decreasing rearranged Fourier series", Annals of Mathematics **144** (1), 167-180 (1996)

[18] T.W. Körner, "Decreasing rearranged Fourier series", The J. Fourier Analysis and Applications **5** (1), 1-19 (1999)

[19] T.W. Körner, "Hard summation, Olevskii, Tao and Walsh", Bull. London Math. Soc. **38**, 705-729 (2006)

[20] S. Kostyukovsky, A. Olevskii, "Note on decreasing rearrangement of Fourier series", J. of Applied Analysis **3** (1), 137-142 (1997)

[21] M. Nielsen, "An example of an almost greedy uniformly bounded orthonormal basis for $L_p(0,1)$", J. Approx. Theory, **149** (2), 188-192 (2007)

[22] T. Tao, "On the almost everywhere convergence of wavelet summation methods" , Appl. Comput. Harmon. Anal. 3, 384-387 (1996)

[23] V.N. Temlyakov, "Nonlinear methods of approximation", Foundations of Computational Mathematics **3** (1), 33-107 (2003)

[24] P. Wojtaszczyk, "Greedy algorithms for general biorthogonal systems", J. Approx. Theory **107** (2), 293-314 (2000)

Yerevan State University, Alex Manoogian 1, Yerevan 0025, Armenia

E-mail address: hayk.aleksanyan@gmail.com