K-SURFACES WITH FREE BOUNDARIES

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ABSTRACT. A well-known question in classical differential geometry and geometric analysis asks for a description of possible boundaries of K-surfaces, which are smooth, compact hypersurfaces in $\mathbb{R}^d$ having constant Gauss curvature equal to $K \geq 0$. This question generated a considerable amount of remarkable results in the last few decades. Motivated by these developments here we study the question of determining a K-surface when only part of its boundary is fixed, and in addition the surface hits a given manifold at some fixed angle. While this general setting is out of reach for us at the present, we settle a model case of the problem, which in its analytic formulation reduces to a Bernoulli type free boundary problem for the Monge-Ampère equation. We study both the cases of 0-curvature and of positive curvature. The formulation of the free boundary condition and its regularity are the most delicate and challenging questions addressed in this work. In this regard we introduce a notion of a *Blaschke extension* of a solution which might be of independent interest.

The problem we study can also be interpreted as the Alt-Caffarelli problem for the Monge-Ampère equation. Moreover, it also relates to the problem of isometric embedding of a positive metric on the annulus with partially prescribed boundary and optimal transport with free mass.

Keywords: K-surface, Gauss curvature, Monge-Ampère equation, Blaschke's rolling ball, Bernoulli free boundary, ruled surface

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1. Introduction

1.1. **Background and motivation.** K-surface in $\mathbb{R}^{d+1}$ ($d \geq 2$) is a smooth compact hypersurface of constant Gauss curvature K. A problem of fundamental importance in classical differential geometry and geometric analysis concerns description of possible boundaries of K-surfaces. Precisely, when a collection $\gamma = \{\gamma_1, ..., \gamma_n\}$ of disjoint $(d-1)$-dimensional closed smooth embedded submanifolds can form a boundary of a (in general immersed) K-surface[1] in $\mathbb{R}^{d+1}$? In particular, S.-T. Yau [25, Problem 26] in his famous list of open problems asks for conditions on a space curve $\gamma \subset \mathbb{R}^3$ to be the image of the boundary of an isometric embedding of the given smooth metric with positive curvature on the disk. In their analytic formulation, locally these problems mostly reduce to equations of Monge-Ampère type. Moreover, for the case of positive curvature $K > 0$, the analysis is restricted to the case of elliptic equations. This drives the study to the class of *locally strictly convex* hypersurfaces[2]. With this in mind, an immediate necessary condition on γ to bound a locally strictly convex hypersurface, is that its second fundamental form must be everywhere non-degenerate (in particular, in $\mathbb{R}^3$ this means that the curve γ is free of inflection points). This elementary limitation, however, is not sufficient. It was shown by Rosenberg [20] that there are further topological objections on γ, which was later demonstrated by Gluck and Pan [10] to be not sufficient either. Most recently, Ghomi [7], answering a question of Rosenberg [20] from 1993, proved that the torsion of any closed space curve bounding a simply connected locally convex surface vanishes at least 4 times. This important result is the only new necessary condition for the existence of K-surfaces since the condition obtained by Rosenberg in early 90's involving the self-linking number of the curve. We give a short review of the literature in subsection 1.3 below.

For many years these problems have served as an important bridge between differential geometry and the analysis of partial differential equations, in particular for equations of Monge-Ampère type, where the advance on either side motivated and lead to new discoveries on the other end. Inspired by these developments, in this paper we introduce and study a class of K-surfaces with free boundaries. Namely, instead of prescribing a set γ of closed smooth strictly convex codimension 2 submanifolds, and asking for a (locally) convex smooth manifold spanning γ, we treat the given γ as part of the boundary of the sought-for surface. Then, fixing a smooth embedded submanifold γ of codimension 1, we ask if there is a K-surface spanning γ and hitting the (given) target manifold T_0 at some prescribed constant angle. In this generality, the problem seems to be out of reach, and here we will study a special case when the boundary of the surface lies in a hyperplane, and the target manifold is a hyperplane parallel to the one containing the portion of the fixed boundary. This basic geometric setting already presents a non-trivial challenge, and seems to be an important model case treated for instance in [16] for the case of classical K-surfaces discussed above. We carry out the analysis in two distinct cases,

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[1] Note, that a K-surface without a boundary with $K > 0$, is a boundary of a convex body.

[2] These surfaces locally lie on one side of their tangent hyperplanes, but need not do so globally.

namely when the curvature is vanishing, and when the curvature is strictly positive. We examine qualitative properties of the problem, i.e. existence, uniqueness and regularity of solutions, as well as regularity and geometric properties of associated free boundaries. Our methods are of both geometric and analytic nature, drawing ideas from convex geometry, theory of Monge-Ampère equations, and free boundary problems.

1.2. **Formal setting and main results.** We now proceed to formal definitions. Fix $\gamma = \{\gamma_1, ..., \gamma_n\}$ a collection of disjoint $(d-1)$-dimensional closed smooth embedded submanifolds in $\mathbb{R}^{d+1}$, and let T_0 be a smooth embedded manifold of codimension 1 in $\mathbb{R}^{d+1}$, and $\lambda_0 > 0$ be a fixed constant. The general question, we have in mind, which we refer to as *K-surfaces with free boundaries*, is the following:

When does there exist a K-surface which spans γ and hits T_0 at the prescribed constant angle $\theta > 0$?

This is of course a generalization of the classical question on K-surfaces to allow free boundaries. A model of this problem, which we will study here, is when the target manifold $T_0 = \mathbb{R}^d \times \{0\}$ and $\gamma \subset \mathbb{R}^d \times \{h_0\}$ is the boundary of some bounded convex set, and $h_0 > 0$ is a given constant. Then, one asks for the existence of a K-surface, having γ on its boundary and intersecting T_0 at a prescribed constant angle θ. In analytic terms, we ask for solvability of the following Bernoulli type free boundary problem for the Monge-Ampère equation: given a convex domain $\Omega \subset \mathbb{R}^d \times \{0\}$ and parameters $h_0, \lambda_0 > 0$, $K_0 \geq 0$ find a concave function $u : \mathbb{R}^d \times \{0\} \to \mathbb{R}_+$ such that

$$\begin{cases} \det D^2(-u) = K_0 \psi(|\nabla u|), & \text{in } \big(\mathbb{R}^d \setminus \overline{\Omega}\big) \cap \{u > 0\}, \\ u = h_0, & \text{on } \partial\Omega, \\ |\nabla u| = \lambda_0, & \text{on } \Gamma_u, \end{cases} \tag{1.1}$$

where $\psi > 0$ is a prescribed real-valued C^∞ function, $\Gamma_u = \partial\{u > 0\} \setminus \overline{\Omega}$ is the free boundary, and the gradient condition is clarified below. The gradient value λ_0 corresponds to the hitting angle $\theta = \arccos \frac{1}{\sqrt{1+\lambda_0^2}}$, as will be clear later. The three main choices for ψ, which are of geometric interest, include $\psi(\xi) = 0$, $\psi(\xi) = 1$, and $\psi(\xi) = (1 + |\xi|^2)^{\frac{d+2}{2}}$, corresponding to the cases of *zero curvature*, *positive curvature measure*, and *positive constant Gauss curvature*, respectively. The latter case for ψ defines the K-hypersurface.

We remark that a problem (1.1) for p-Laplace operator was studied by Henrot and Shahgholian [15]. Although the analytic formulation of the problem is similar here, our motivation and methods are entirely different from those of [15].

1.2.1. *The free boundary condition.* The gradient condition on boundary in (1.1) is understood in a weak sense. Namely, whenever at $x_0 \in \Gamma_u$ there is a well-defined unit inner normal to the set Γ_u, then the normal derivative of u at x_0 equals λ_0. This is in view of a standard fact that concave (convex) functions have directional derivatives. Next, we observe that the convexity of the set Γ_u implies that it has well-defined normal almost everywhere (with respect to the surface measure), and hence in (1.1) the free boundary condition must hold almost everywhere.

It will be convenient for us, to work with a geometric reformulation of the last requirement of (1.1), in terms of the *slope* of a support hyperplane, to which we now proceed. Consider the convex body $\mathcal{K}$ bounded by the graph of u and hyperplanes $\mathbb{R}^d \times \{0\}$ and $\mathbb{R}^d \times \{h_0\}$. Let also $X_0 = (x_0, 0) \in \mathbb{R}^d \times \mathbb{R}$ be a point on Γ_u where the normal to the free boundary exists. Now let $H = \{X \in \mathbb{R}^{d+1} : \ (X - X_0) \cdot \nu = 0\}$ be a support hyperplane to $\mathcal{K}$ at X_0, where $\nu = (\nu_1, ..., \nu_{d+1}) \in \mathbb{R}^{d+1}$ is a unit vector. Clearly the hyperplane $G := H \cap (\mathbb{R}^d \times \{0\})$ defines a support plane for Γ_u through x_0, and due to the existence of a normal at x_0, G is the unique support hyperplane to Γ_u through x_0. We conclude that H has only one degree of freedom, determined by the angle it makes with $\mathbb{R}^d \times \{0\}$. From now on, we will only consider the set of those H which are above Ω, i.e. the last coordinate of their unit normal is negative. Using the fact $|\nu_{d+1}| < 1$ we define $\nu_* := \frac{1}{(1-\nu_{d+1}^2)^{1/2}} \overline{\nu}$ where $\overline{\nu} \in \mathbb{R}^d$ is formed from the first d coordinates of ν. By definition $|\nu_*| = 1$, and by construction the linear function $H(x) := -\frac{\overline{\nu}}{\nu_{d+1}} \cdot (x - x_0)$ bounds u from above, and in particular $\nu_{d+1} < 0$. Note that if ν is the unit normal to the support function at a free boundary point x_0 with $\nu_{d+1} < 0$ then ν_* is the inner unit normal at x_0. We now fix the following:

Definition 1.1. *For any hyperplane* $H = \{X \in \mathbb{R}^{d+1} : \ (X - X_0) \cdot \nu = 0\}$, *where* $\nu = (\nu_1, ..., \nu_{d+1})$ *is a unit vector, the vector* $-\frac{\overline{\nu}}{\nu_{d+1}}$ *is called the slope of* H.

Remark 1.1. *When it will be clear from the context, we will abuse the notation, and refer to the length of the slope, as the slope of a hyperplane. Getting back to the last condition in* (1.1), *we conclude that at any point* $x_0 \in \Gamma_u$, *having a well-defined unit inward normal* ν, *the equivalent definition of the free boundary condition* (1.1) *is that among all support planes to the graph of* u *at* x_0 *and staying above* Ω, *the smallest slope has length* λ_0. *This is easy to check, using the discussion above. Geometrically, it means that the hyperplane* H *cannot be inclined on the graph of* u *more than the angle* $\arccos(1 + \lambda_0^2)^{-1/2}$. *Such support functions will be referred to as* **extreme**.

1.2.2. *Weak solutions.* Here we define the notion of a weak solution (after A.D. Aleksandrov) to (1.1) and recall some key concepts related to the Monge-Ampère operator.

Definition 1.2. *Let* u *be a convex function defined in some domain* $D \subset \mathbb{R}^d$. *Given any* $x_0 \in D$, *the following set of slopes*

$$\omega_{x_0}(u) = \{p \in \mathbb{R}^d : \quad u(x) \geq p \cdot (x - x_0) + u(x_0) \quad \forall x \in D\},$$

is called the gradient mapping of u *at* x_0; *correspondingly for a set* $E \subset \Omega$ *we set*

$$\omega_E(u) = \bigcup_{x \in E} \omega_x(u).$$

Due to Aleksandrov's theorem ω_E is measurable for any Borel set $E \subset D$ and induces a Borel measure on D by the following way

$$\mu(E) = |\omega_E(u)|, \tag{1.2}$$

where $E \subset D$ is a Borel set, and $|\cdot|$ stands for the Lebesgue measure. The measure μ defined by (1.2) is called the Monge-Ampère measure. In particular, one has the following fundamental property: the set of all slopes $p \in \mathbb{R}^d$ which belong to the gradient image of more than one point of D has Lebesgue measure 0 (see [1, p. 190], also [19, Theorem 2.5], and [14]).

An important property of the Monge-Ampère measure is its weak continuity in the following sense: if $\{v_m\}$ is a sequence of convex functions on D such that $v_m \to v$ locally uniformly in D then the measure $\mu_m(E) = |\chi_{v_m}(E)|$ converges weakly to $\mu(E) = |\chi_v(E)|$, see [23], [14]. This is also true for the measure

$$\mu_\psi(E) = \int_{\omega_E(u)} \frac{d\xi}{\psi(\xi)}$$

see [17].

We next define the solution to (1.1).

Definition 1.3. *We say that a concave function $u : \mathbb{R}^d \setminus \Omega \to \mathbb{R}$ is a weak solution (corr. super-solution) to* (1.1) *if the following hold:*

(a) *for any Borel set $E \subset \{u > 0\}$ we have*

$$\int_{\omega_E(-u)} \frac{d\xi}{\psi(\xi)} \ (\geq) \ = K_0|E|,$$

where $\omega_E(-u)$ is the gradient image of $-u$ on E,

(b) *$u = h_0$ on $\partial\Omega$ pointwise,*

(c) *for any point $x_0 \in \partial\{u > 0\} \setminus \overline{\Omega} =: \Gamma_u$ on the free boundary where the set Γ_u has a well-defined normal, and among the all support hyperplanes to the graph of u at x_0 the smallest slope equals (is bounded above by) λ_0. The class of super-solutions is denoted by $\mathbb{W}_+(K_0, \lambda_0, \Omega)$.*

The last condition (c) with slopes should be understood in a sense discussed in subsection 1.2.1. Also, when the constant $K_0 = 0$ in (1.1), the condition (a) above, means that the Monge-Ampère measure associated with u is trivial.

1.2.3. *Main results.*

The following are our main results.

Theorem A. *In* (1.1) *take $K_0 = 0$ and assume Ω is a bounded convex domain, which is $C^{1,1}$-regular. Then, there exists a unique concave function u which is a weak solution* (1.1). *Moreover, the solution u is a ruled surface, which is $C^{1,1}$ in the set $\{u > 0\} \setminus \overline{\Omega}$, and the free boundary Γ_u is $C^{1,1}$ also. If in addition Ω is strictly convex, then so is the free boundary.*

Remark 1.2. *The methods of our paper allow for the following (easy) generalization. Instead of the hyperplane $\mathbb{R}^d \times \{h_0\}$, where the boundary of the surface was prescribed, we can consider a hyperplane of the form*

$$\Sigma = \{x \in \mathbb{R}^{d+1} : \ x_{d+1} = l_1 x_1 + ... + l_d x_d + a\}, \quad l = (l_1, ..., l_d) \in \mathbb{R}^d,$$

and then let γ–the boundary of the surface (i.e. the boundary data of u in the problem 1.1*), to be equal to the intersection of the cylinder $\partial\Omega \times \mathbb{R}$ and Σ. Here we should only require that Ω and Σ are chosen so that γ does not intersect the* *target hyperplane* $T_0 = \mathbb{R}^d \times \{0\}$. *Next, assuming the compatibility condition that* $\sum\limits_{i=1}^{d} l_i^2 \leq \lambda_0^2$, *i.e. the target hyperplane can be seen from* Σ *at a contact angle at least* $\arccos(1+\lambda_0)^{-1/2}$, *we can still work out the details of our arguments.*

Theorem B. *Let* Ω *be uniformly strictly convex and* C^2 *regular. Let* $\psi(\xi) = \psi(|\xi|)$ *be* C^∞ *smooth nondecreasing positive function. Then, there is a universal constant* $K = K(\lambda_0, \Omega, \psi) > 0$ *small such that if* $K_0 \in (0, K)$ *then* (1.1) *has a weak solution* u, *which is* C^∞ *in* $\{u > 0\} \setminus \overline{\Omega}$ *and the free boundary* Γ_u *is* C^∞ *regular.*

The smallness assumption on K_0 cannot be eliminated entirely as indicated by explicit computations for the radially symmetric solutions, see the Appendix.

We prove Theorem A in Section 2, and Theorem B in Section 3.

Remark 1.3. *When* $\psi(t) = 1$, *in* (1.1) *we get the equation for prescribed curvature measure, and for* $\psi(t) = (1+t^2)^{(d+2)/2}$ *we recover the case of prescribed Gauss curvature. In both cases condition* (3.1), *which reads*

$$K_0^{\frac{1}{d}} \leq \psi^{-\frac{1}{d}}(\lambda_0) \frac{\kappa_0 \lambda_0^2}{h_0 \kappa_0 + (\lambda_0^2 + h_0^2 \kappa_0^2)^{1/2}},$$

where κ_0 *is the smallest curvature of* $\partial\Omega$, *is sufficient for the existence of a solution, and it is in the regularity of the free boundary that we need to make* K_0 *even smaller. We also remark that for* $\psi = 1$ *the case of equality in the above inequality coincides, as one would expect, with the identity in* (A.8) *concerning the radially symmetric solutions.*

Computations similar to those we did in subsection A.2 show that for general parameters involved (1.1) *there can be no solution for the prescribed Gauss curvature equation. The computations, while tedious, present no difficulty, and we will leave it out for the interested reader to explore.*

We also remark that the monotonicity of ψ *is being used to get a neat formulation for the free boundary condition. One should be able to treat more general cases for* ψ, *but here we do not attempt to full generality, and rather focus on model of the problem.*

Notation. The following notation will be in force throughout the paper.

$C, C_0, C_n, \ldots$	positive constants, that can vary from formula to formula
∂U	the boundary of a set U,
Σ	the hyperplane $\mathbb{R}^d \times \{h_0\} \subset \mathbb{R}^{d+1}$,
$\widehat{\Omega}$	$(\Omega \times \mathbb{R}) \cap \Sigma$,
$B(x, r)$	the d-dimensional Euclidean ball of radius $r > 0$ and center at x,
$\Gamma_u = \{u > 0\} \setminus \overline{\Omega}$	the free boundary,
$\mathrm{Hull}(E)$	the convex hull of a set $E \subset \mathbb{R}^d$,
ν	the unit inner normal,
T_0	the target manifold $\mathbb{R}^d \times \{0\}$,
$\mathbb{W}_+(K_0, \lambda_0, d)$	the class of super-solutions, see Definition 1.3.

1.3. **Related works.** We present a short review of the theory of K-surfaces. The literature quoted below is by no means exhaustive, but is selected, slightly arbitrarily, with a hope to give the reader a flavour of the breadth and scopes of this area of research.

One may roughly categorise the results on existence of K-surfaces in two groups; namely, those where the sought-for K-surface is globally a graph of a function over some (nice) domain, and those where the surface is not necessarily a graph globally (this setting allows immersed surfaces too, in particular). The main advantage of the first scenario, is the availability of a global coordinate system, where the existence of the surface in question, can be reformulated in terms of some boundary value problem for Monge-Ampère equation. Then, having some nice control over the geometry of the domain where the problem is posed, the original geometric problem is being treated as a question in the existence theory for Monge-Ampère equations. In this direction, the results of Caffarelli, Nirenberg, and Spruck [5], show that if $\Omega \subset \mathbb{R}^d$ is a bounded, smooth and a strictly convex domain, then for any smooth function φ over $\partial\Omega$, if $K > 0$ is small enough, then there is a K-surface spanning the graph of φ. This existence result was extended by Hoffman, Rosenberg, and Spruck [16] to the case of boundaries with multiple components, where among other results, it was proved that given any two strictly convex curves lying in two parallel hyperplanes, and such that one of the curves can be moved inside the other by a translation, then for a sufficiently small $K > 0$ there exists a K-surface having these curves as its boundary. Both of these papers treat the existence of K-surfaces primarily through the existence theory of Monge-Ampère equations. Later on, more general settings were treated by Guan and Spruck [11], and Guan [13]. One of the results from [11], which is particularly striking, proves the existence of locally convex embedded K-surfaces with arbitrary high genus. This remarkable result shows that the class of K-surfaces can be very complicated in spite of strict conditions on its geometry.

We now move to the second category of the existence results, where the surface is not necessarily a graph. In this direction, the paper of Guan and Spruck [12], establishes a rather general existence result. Namely, assuming the existence of a locally convex immersed and C^2-smooth hypersurface $\Sigma \subset \mathbb{R}^{d+1}$, which is locally strictly convex along its boundary, it is proved that for any positive K which nowhere exceeds the curvature of Σ, there exists an up to the boundary smooth, locally strictly convex immersed hypersurface M having $\partial\Sigma$ as its boundary and having curvature K everywhere. This result was also independently obtained by Trudinger and Wang [22]. Finally, let us quote the important paper by Ghomi [9] which brings direct geometrical and topological ideas into the existence theory of K-surfaces.

In the last part of our review we note that the all cases discussed above, treated the case of strictly positive curvature. For the vanishing curvature case, which comprises another important class of problems, it is proved by Guan and Spruck [12] that if γ bounds a locally strictly convex codimension 2 hypersurface, which is

C^2-smooth, there exists a locally convex $C^{1,1}$-smooth hypersurface M having γ as its boundary.

1.4. **Optimal transport with free boundary between annuli.** The problem (1.1) can be regarded as the analogue of Alt-Caffarelli problem [2] for the Monge-Ampère equation.

Another interesting interpretation can be given in terms of optimal mass transport theory. Suppose $\psi = 1$, $0 \in \Omega$ and let $w = u + \frac{1}{2}|x|^2$. Then $-w$ is a potential (modulo a constant summand) defining the transport map $y = x - \nabla w$ which maps the annulus bounded by Γ_u and $\partial\Omega$ to the one bounded by $\partial B(0, \lambda_0)$ and $\nabla u(\Gamma_u)$, see Figure 1.

Moreover, w solves the following overdetermined problem for the unknown w:

$$\begin{cases} \det(-D^2 w + \mathrm{Id}) = K_0, & \text{in } \left(\mathbb{R}^d \setminus \overline{\Omega}\right) \cap \{w > \frac{1}{2}|x|^2\}, \\ w = h_0 + \frac{1}{2}|x|^2, & \text{on } \partial\Omega, \\ |\nabla w - x| = \lambda_0, & \text{on } w = \frac{1}{2}|x|^2. \end{cases} \tag{1.3}$$

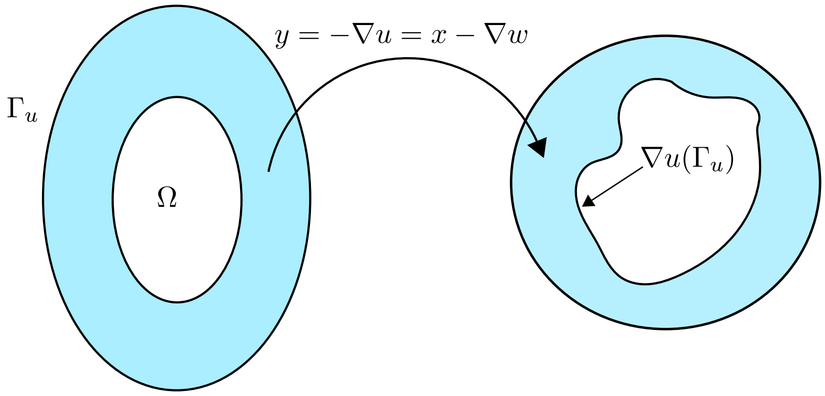

Figure 1. The annular domains having one free and one prescribed component represent the reference and target sets under the transport map $y = -\nabla u$.

The total mass of transport m is also unknown and must be determined from w by the formula

$$m := \mathrm{Vol}\left(\{u > 0\} \setminus \Omega\right) = \frac{1}{K_0} \mathrm{Vol}\left(\nabla(\{u > 0\} \setminus \Omega)\right).$$

To the best of our knowledge the study of these type of problems does not appear in the existing literature.

2. The homogeneous problem

In this section we prove Theorem A which concerns (1.1) for $K_0 = 0$, and this assumption will be in force throughout the section in any reference to (1.1) unless explicitly stated otherwise. The free boundary condition in Definition 1.3 makes the problem ill-posed. As the example for Ω being a triangle in $\mathbb{R}^n$ shows, for the same data we may have more than one solution. Moreover, one of them is $C^{1,1}$ smooth while the other is only Lipschitz continuous, see Figure 2.1. In Lemma 2.2 we show the existence of $C^{1,1}$ solution using the Minkowski sum of Ω with a ball of suitable radius to construct the free boundary. The existence of a weak, not necessarily C^1 solution is proved in Proposition 2.5.

We start with a simple example, which is meant to illustrate how the solution to (1.1) looks like in the simplest case.

Example 2.1. (Truncated cones) *Let Ω be the unit ball of $\mathbb{R}^d$ ($d \geq 2$) and fix $\lambda > 0$. Consider the problem (1.1) with $K_0 = 0$, gradient condition $\lambda > 0$, and boundary data identically 1. Then, we can easily see (by direct computation) that the function $u(x) = 1 + \lambda - \lambda|x|$, with $1 \leq |x| \leq 1 + 1/\lambda$ defines a solution to (1.1) which is concave on its positivity set (with a convention that u is extended as identically 1 in Ω). Clearly the free boundary is the circle $|x| = 1 + 1/\lambda$ and the free boundary condition in (1.1) is satisfied at all points. Note that the graph of u is a boundary of a solid truncated cone.*

What we can also observe from this example, is that at any point of the free boundary, there is a unique support plane to the graph of u having slope λ. More precisely, for any $M = (x_1, x_2, ..., x_d)$ on the free boundary, the unit normal $\nu \in \mathbb{S}^d$ of this hyperplane has the form

$$\nu = \frac{1}{(1+\lambda^2)^{1/2}} \left(-\frac{\lambda^2}{1+\lambda} x_1, ..., -\frac{\lambda^2}{1+\lambda} x_d, -1 \right) \in \mathbb{S}^d.$$

It is also clear that for any ball $B(x_0, r)$, where $x_0 \in \mathbb{R}^d$ and $r > 0$, we can translate and scale u constructed for $B(0,1)$ to get a conical solution for a general ball too. Namely, setting

$$v(x) = 1 + \lambda - \frac{\lambda}{r}|x - x_0|, \quad \text{where } r \leq |x - x_0| \leq r\left(1 + \frac{1}{\lambda}\right),$$

we get a solution to (1.1) for $\Omega = B(x_0, r)$, $K_0 = 0$, and the given $\lambda > 0$. The free boundary in this case will be the sphere $|x - x_0| = r\left(1 + \frac{1}{\lambda}\right)$.

Finally, observe that a similar construction provides a convex solution to (1.1) with the same data, however with the free boundary contained inside Ω.

2.1. **Existence of regular solution.**

Lemma 2.2. *Let $\partial\Omega$ be convex. Then there is a $C^{1,1}$ solution such that the free boundary is the boundary of the Minkowski sum $\Omega + a\nu$ where ν is the exterior unit normal to $\partial\Omega$ and $a = h_0/\tan\theta$ and $\theta = \arccos \frac{1}{\sqrt{1+\lambda_0^2}}$. Moreover, the free boundary is $C^{1,1}$.*

Proof. Consider the function

$$u(x) = h_0 - \operatorname{dist}(x, \Omega) \tan\theta,$$

clearly u is concave. Let $\Gamma = \{y : \operatorname{dist}(y, \Omega) = \frac{h_0}{\tan\theta}\}$ and $y_0 \in \partial\Omega$ such that $|y - y_0| = \operatorname{dist}(y, \Omega)$, then for $y_t = yt + (1-t)y_0, t \in (0,1)$ we have that

$$u(y_t) = h_0 - |y - y_0|(1-t)\tan\theta \tag{2.1}$$

which is a line and hence the graph of u is ruled. It follows that the Gauss curvature of the graph of u vanishes everywhere.

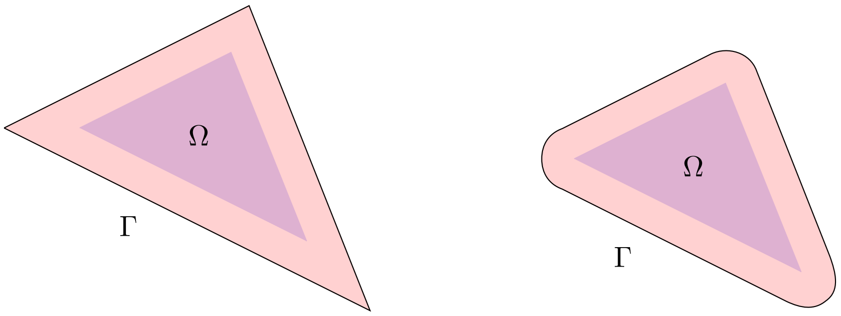

FIGURE 2. If Ω is a triangle in $\mathbb{R}^2$ then there are two solutions verifying the free boundary condition almost everywhere on Γ. The red region is $\{u > 0\}$.

Let $\Gamma = \partial\{u > 0\} \setminus \overline{\Omega}$ be the free boundary. From the definition of u we have that Γ is the $\frac{h_0}{\tan\theta}$-level surface of the distance function from Ω. But then, Γ is the boundary of the outer parallel set of Ω at distance $\frac{h_0}{\tan\theta}$, i.e. the boundary of the set $\Omega + \frac{h_0}{\tan\theta} B(0,1)$, where $B(0,1)$ is the unit ball of $\mathbb{R}^d$ at the origin, and the sum is in the Minkowski sense. In view of this at each point of Γ there is an interior touching ball of radius $\frac{h_0}{\tan\theta}$, and hence Γ is uniformly $C^{1,1}$.

Next we show that u is $C^{1,1}$ up to the boundary. As we saw above each level-surface of u is $C^{1,1}$ uniformly if we stay away from Ω. In view of (2.1) through each point $X = (x_0, u(x_0)) \in \mathbb{R}^{d+1}$ on the graph of u there is a line segment lying on the graph of u. Call this segment I. The argument of the previous paragraph shows that through each point $(x, u(x))$ of I there is a ball B with radius bounded below by a constant multiple of a distance of x from Ω, such that B lies in the set $\{x \in \mathbb{R}^d : u(x) = u(x_0)\} \times \{u(x_0)\}$. Sliding these balls across I we get a cylinder sharing a line segment with the graph of u and lying below the surface of u. Hence at each point of the graph of u there is a $(d+1)$-dimensional ball touching the graph of u at the given point and staying completely under the graph of u. This proves that u is $C^{1,1}$ up to Γ and completes the proof of the current lemma.

□

2.2. **The case of polygonal domains.** As was discussed above, problem (1.1) with $K_0 = 0$ is ill-posed. Lemma 2.2 shows that for any convex Ω there is a $C^{1,1}$ solution to (1.1) with $K_0 = 0$, while from Figure 2.1 we see that this solution in general is not unique. Here we present an alternative approach to the existence of solution in 0-curvature case based on approximation argument by polyhedrons, which not always coincides with the solution given by Lemma 2.2.

For the sake of notation we identify $\Omega \subset \mathbb{R}^d$ with $\Omega \times \{0\} \subset \mathbb{R}^{d+1}$.

Proposition 2.3. *In* (1.1) *let* Ω *be a convex polyhedron. Then there is a generalized solution to* (1.1).

Proof. Set $\Sigma = \{X \in \mathbb{R}^{d+1} : \ x_{d+1} = h_0\}$, and by $F_1, ..., F_n$ denote the facets of Ω, i.e. the $(d-1)$-dimensional flat portions of $\partial\Omega$. For the orthogonal projection operator in the $(d+1)$-th direction

$$\pi_{d+1}(X) = (x_1, ..., x_d, 0), \text{ where } X = (x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1}$$

denote $\widehat{\Omega} = \pi_{d+1}^{-1}(\Omega) \cap \Sigma$, i.e. the lift of Ω into Σ. Similarly, for each $1 \leq i \leq n$ define $G_i = \pi_{d+1}^{-1}(F_i) \cap \Sigma$. It is clear that $\widehat{\Omega}$ is a convex polyhedron in Σ with facets $G_1, ..., G_n$.

For $1 \leq i \leq n$ consider a hyperplane $H_i = \{X \in \mathbb{R}^{d+1} : \ X \cdot \nu^{(i)} = c_i\}$ in $\mathbb{R}^{d+1}$, where $\nu^{(i)}$ is the unit normal and $c_i \in \mathbb{R}$. We choose H_i so that it passes through the facet G_i, has unit normal $\nu^{(i)}$ satisfying

$$\nu_{d+1}^{(i)} = (1 + \lambda_0^2)^{-1/2} \tag{2.2}$$

and for its graph

$$u_i(x) = \frac{1}{\nu_{d+1}^{(i)}} \left[c_i - (\nu_1^{(i)} x_1 + ... + \nu_d^{(i)} x_d) \right], \qquad x \in \mathbb{R}^d$$

we have

$$\Omega \subset \{(x, 0) \in \mathbb{R}^d \times \{0\} : \ u_i(x) \geq h_0\}.$$

It is clear that such H_i exists. Since $|\nu^{(i)}| = 1$ from (2.2) we get

$$|\nabla u_i(x)| = \frac{\left(1 - (\nu_{d+1}^{(i)})^2\right)^{1/2}}{\nu_{d+1}^{(i)}} = \lambda_0 \text{ on } \mathbb{R}^d. \tag{2.3}$$

We now define

$$\mathcal{U} = \bigcap_{i=1}^{n} \{x \in \mathbb{R}^d : \ u_i(x) \geq 0\} \setminus \Omega.$$

Observe that $\mathcal{U}$ defines a bounded set in $\mathbb{R}^d$. For $x \in \mathcal{U}$ consider the lower envelope

$$u(x) = \min_{1 \leq i \leq n} u_i(x). \tag{2.4}$$

We thus have that u is a piecewise linear function. Moreover, it follows by (2.3) that $|\nabla u(x)| = \lambda_0$ on $\partial\{x \in \mathbb{R}^d : u(x) > 0\} \setminus \overline{\Omega}$ away from a set of zero $\mathcal{H}^{d-1}$-measure (namely, away from the non-smooth boundary of the polyhedron $\{x \in \mathbb{R}^d : \ u(x) = 0\}$). We also have $u = h_0$ on $\partial\Omega$ by construction. Thus it is only left to prove that u is a solution to the equation. To this end define $\mathcal{K}$ to be the body bounded by the graph of u and hyperplanes $x_{d+1} = 0$ and Σ. More precisely,

$$\mathcal{K} = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : \ x \in \overline{\mathcal{U}}, \ 0 \leq t \leq u(x)\} \cup \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : x \in \overline{\Omega}, \ 0 \leq t \leq h_0\}.$$

By construction $\mathcal{K}$ is a convex polyhedron in $\mathbb{R}^{d+1}$. Since u is non-smooth, to show $\det D^2(-u) = 0$ in $\{u > 0\} \setminus \overline{\Omega}$, we need to prove that the corresponding Monge-Ampère measure is vanishing (see subsection 1.2.2), or equivalently that the gradient image of u on its positivity set has measure 0. To see this, observe that at points where u is smooth, its gradient mapping assumes only finitely many values, which are precisely the gradients of u_i-s considered above. We are thus left to treat the points where u is not differentiable. It is clear that u is non-smooth at some $x \in \mathcal{U}$ iff $X = (x, u(x))$ lies on the non-smooth boundary of the polyhedron $\mathcal{K}$. But the non-smooth boundary of the polyhedron is the union of its k-dimensional edges, where $k = 0, ..., d-1$. Along an edge $\mathcal{E}$ of dimension $k \geq 1$, the gradient mapping of u is the same, namely for any x, y in the interior of $\mathcal{E}$, we have $\omega_x(u) = \omega_y(u)$. Hence, any $\alpha \in \omega_{\mathcal{E}}(u)$ lies in the gradient image of at least two different points, and therefore by the celebrated result of Aleksandrov (see subsection 1.2.2) we get that $|\omega_{\mathcal{E}}(u)| = 0$. As a result we see that in order to complete the proof, we need to show that $\mathcal{K}$ has no edges of dimension 0 (i.e. vertices) in the strip $0 < x_{d+1} < h_0$. In what follows we prove this statement and hence the proposition. For the sake of clarity we split the proof into few steps.

Step 1. *Vertex of $\mathcal{K}$ and a ray through it.*

Assume for contradiction, that $X^{(0)} = (x^{(0)}, u(x^{(0)})) \in \mathbb{R}^{d+1}$ is a point on the graph of u satisfying $0 < X^{(0)} \cdot e_{d+1} < h_0$ and such that there is no line on the graph of u passing through $X^{(0)}$. Then there must be at least $d+1$ hyperplanes H_i having a unique intersection point at $X^{(0)}$. Consequently, the system of linear equations

$$(X - X^{(0)}) \cdot \nu^{(i)} = 0 \quad \text{for all} \quad i = 1, ..., d+1, \tag{2.5}$$

has a unique solution, hence the collection $\{\nu^{(1)}, ..., \nu^{(d+1)}\}$ is linearly independent in $\mathbb{R}^{d+1}$. This implies, that any d out of these $d+1$ hyperplanes intersect in a line (e.g. by rank-nullity theorem). We claim that at least one of these $d+1$ lines must intersect Σ. Indeed, if not the case, then all $d+1$ lines must be parallel to Σ, and hence if l_i is the direction vector of the i-th line, then $l_i \cdot e_{d+1} = 0$ for all $1 \leq i \leq d+1$, where e_{d+1} is the normal vector of Σ. We have that $\{\nu^{(i)}\}_{i=1}^{d+1}$ is linearly independent, and e_{d+1} is a non-zero vector. Hence, we may replace one of $\nu^{(i)}$-s by e_{d+1} and still get a collection of $(d+1)$-linearly independent vectors. With this in mind, assume that $\{\nu^{(1)}, ..., \nu^{(d)}, e_{d+1}\}$ is linearly independent, and let the line L_1 be the intersection of H_1,...,H_d. By assumption, for l_1, the direction vector of L_1, we have $l_1 \cdot \nu^{(i)} = 0$ for all $1 \leq i \leq d$ since $L_1 \subset H_i$. As L_1 does not intersect Σ, it follows that $l_1 \cdot e_{d+1} = 0$. We get that l_1 is orthogonal to $d+1$ linearly independent vectors, which is a contradiction since l_1 is non-zero. Hence, we conclude that at least one of the lines through $X^{(0)}$ formed as an intersection of d planes must intersect Σ. Let L_0 be the ray on this line intersecting Σ.

Step 2. *Projection of L_0.*

Recall that each H_i passes through the facet G_i by construction. At this stage we identify G_i with its bounding $(d-1)$-dimensional hyperplane in Σ; clearly $G_i = H_i \cap \Sigma$. Let $\widehat{L}_0$ be the orthogonal projection of L_0 onto Σ. We claim that each point on $\widehat{L}_0$ is equidistant from all $G_1, ..., G_d$. To see this, we first observe that the equation of H_i gives

$$G_i = \{(x, h_0) \in \mathbb{R}^d \times \mathbb{R} : \; x \cdot \overline{\nu^{(i)}} - \overline{X^{(0)}} \cdot \overline{\nu^{(i)}} + (h_0 - X^{(0)}_{d+1})\nu^{(i)}_{d+1} = 0\}, \tag{2.6}$$

where for $X = (x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1}$ we have set $\overline{X} = (x_1, ..., x_d)$. Now if $Z = (z, z_{d+1}) \in L_0$ is any, then for its projection $\widehat{Z} = (z, h_0) \in \widehat{L}_0$ and for each $1 \leq i \leq d$ we have

$$\operatorname{dist}(\widehat{Z}, G_i) = \frac{|z \cdot \overline{\nu^{(i)}} - \overline{X^{(0)}} \cdot \overline{\nu}^{(i)} + (h_0 - X^{(0)}_{d+1})\nu^{(0)}_{d+1}|}{|\overline{\nu^{(i)}}|} = \frac{|h_0 - z_{d+1}||\nu^{(i)}_{d+1}|}{|\overline{\nu^{(i)}}|}, \tag{2.7}$$

where the second equality is due to the fact that $Z \in L_0 \subset H_i$ for any $1 \le i \le d$, and condition (2.2). Applying (2.2) to (2.7) we get that $\mathrm{dist}(\widehat{Z}, G_i)$ is independent of $1 \le i \le d$ for each given $Z \in L_0$.

Step 3. *Round cones touching the facets.*

Let Y_0 be the intersection of L_0 and Σ. By independence of $\nu^{(1)}, ..., \nu^{(d)}$, the point Y_0 forms a vertex of the polyhedron $\widehat{\Omega}$. In particular we get that $\widehat{L_0} \cap int(\widehat{\Omega}) \neq \emptyset$. Take $\widehat{Z} \in \widehat{L_0} \cap int(\widehat{\Omega})$ close to Y_0. Then by Step 2 we have that the ball $B_Z := B(\widehat{Z}, \mathrm{dist}(\widehat{Z}, G_1))$ touches facets $G_1, ..., G_d$ tangentially. We next consider a round cone $\mathcal{C}_Z$ which has the ball B_Z as its base, and has vertex at $Z \in L_0$ which has $\widehat{Z}$ as its projection. In view of the construction we have that the hyperplanes $H_1, ..., H_d$ are support planes for $\mathcal{C}_Z$ all intersecting the boundary of the cone in line segments. Moving the point Z along the ray L_0, we will get eventually, for some Z, that the ball B_Z touches G_{d+1} as well, thanks to the convexity of the region bounded by facets $G_1, ..., G_{d+1}$. Hence, the cone $\mathcal{C}_Z$ at that position will have H_{d+1} as its support plane too, as being a round cone of slope λ_0, it equals the lower envelope of its all support hyperplanes of slope λ_0. But then, the ray L_0 has to intersect H_{d+1} above Σ, producing a new intersection point of $H_1, ..., H_{d+1}$. The latter contradicts to the uniqueness of $X^{(0)}$, and hence completes the proof of the proposition. ☐

Remark 2.4. *It is worthwhile to observe, that in the case when the all $d+1$ rays intersect Σ (Step 1 in the proof), a different argument handles the proof above. Indeed, in such a case we get that the facets G_1,...,G_{d+1} have the property that any d of them intersect in a single point. Moreover all these $d+1$ points are different, as if any two were the same, that point would lie in the intersection of the hyperplanes H_i but which we know contains a single point, which is $X^{(0)}$. These properties imply that $\widehat{\Omega}$ is a d-simplex. But a d-dimensional simplex has exactly $d+1$ number of different facets of dimension $d-1$; we thus get that the collection $\{G_1, ..., G_{d+1}\}$ exhausts the all facets of $\widehat{\Omega}$. Now the contradiction follows easily since there must be at least one H_i whose graph has value larger than h_0 at $X^{(0)}$. But by assumption all H_i-s are strictly less than h_0 at $X^{(0)}$ and we get a contradiction, by so completing the argument.*

2.3. **Approximation argument.** Here we study the case of general convex domain. Let Ω be a bounded convex domain with C^1 boundary. Thanks to the smoothness of $\partial\Omega$ for each $X_0 \in \partial\widehat{\Omega}$ there exists a unique support hyperplane to $\widehat{\Omega}$ at X_0 having slope λ_0; call this plane H_{X_0}. Following the discussion in subsection 1.2.1, each H_{X_0} can be identified with a linear function over $\mathbb{R}^d$. With this in mind, consider the lower envelope of these support hyperplanes

$$h_*(x) := \inf_{X_0 \in \partial\widehat{\Omega}} H_{X_0}(x), \qquad x \in \mathbb{R}^d. \tag{2.8}$$

The infimum here does not collapse, thanks to the condition on the slopes of support planes, and hence (2.8) defines a function locally bounded below on $\mathbb{R}^d$. Indeed, let us see that the zero set of any H_{X_0} stays on a uniform distance from Ω. Similar to the computations in the proof of Proposition 2.3, we can identify each H_{X_0} having unit normal $\nu \in \mathbb{R}^{d+1}$ with a linear function $H(x) = -\frac{1}{\nu_{d+1}}[x \cdot \overline{\nu} + X_0 \cdot \nu]$, where $x \in \mathbb{R}^d$, and $\nu = (\overline{\nu}, \nu_{d+1})$ with $\nu_{d+1} = -(1+\lambda_0^2)^{1/2}$. Then, if $x \in \overline{\Omega}$ we have $H(x) \geq h_0$ and for any $y \in \mathbb{R}^d$ where $H(y) = 0$ we get

$$h_0(1+\lambda_0^2)^{1/2} \leq (x-y) \cdot \overline{\nu} \leq |x-y||\overline{\nu}|,$$

and hence, h_* gives a locally bounded function which is concave as a lower envelope of concave functions. We have the following result.

Proposition 2.5. *Let Ω be a bounded convex domain in $\mathbb{R}^d$ with C^1 boundary. Then h_* is a weak solution to* (1.1).

Proof. We will approximate h_* by polyhedral solutions, and will use the weak continuity of Monge-Ampère measure. To this end, fix a sequence $\{x^{(n)}\}_{n=1}^{\infty} \subset \partial\Omega$ forming a set of dense distinct points in $\partial\Omega$. For each integer $n \geq n_0$, where $n_0 \in \mathbb{N}$ is large enough, we let Ω_n be the convex polyhedron bounded by tangent hyperplanes of Ω at points $\{x^{(1)}, ..., x^{(n)}\}$; obviously $\Omega \subset \Omega_n$. Define $g_n : \partial\Omega_n \to \mathbb{R}$ by $g_n(x) = \pi_{d+1}^{-1}(x) \cap \Sigma$ where $x \in \partial\Omega_n$. We now let u_n be the solution of (1.1) constructed in Proposition 2.3 for Ω_n and g_n. For notational convenience we extend each u_n into Ω_n having the same graph as the hyperplane Σ, in this case as identically equal to constant h_0; this extension is also denoted by u_n. Observe as well, that by (2.4) each u_n is defined everywhere on $\mathbb{R}^d$. Finally, after relabelling we assume that $n_0 = 1$.

By (2.4) we have

$$u_1(x) \geq ... \geq u_n(x) \geq ... \geq h_*(x) \qquad x \in \mathbb{R}^d. \tag{2.9}$$

Since u_n is decreasing and bounded below, it converges to some function u_0 defined on $\mathbb{R}^d$, which is clearly concave. We claim that the convergence is uniform in the set $\mathfrak{U}_1 := \{u_1 \geq -1\}$; this will follow from a well-known fact that convex functions are locally Lipschitz, with Lipschitz constant admitting a bound by L^∞ norm of the function.

Since the sequence $\{u_n\}$ is uniformly bounded on $\mathfrak{U}_1$, by [3, Lemma 3.1] the set of gradient images of $\{u_n\}$ will be uniformly bounded on the set $\{u_1 \geq 0\}$, which contains the sets $\{u_n \geq 0\}$ by (2.9). This in its turn implies uniform Lipschitz bounds on the sequence $\{u_n\}$ on the set $\{u_1 \geq 0\}$ by [19, Proposition 2.4]. Hence, Arzelà-Ascoli implies existence of a subsequence $\{n_k\}$ along which u_{n_k} converges to u_0 uniformly on the set $\{u_0 \geq 0\}$. But then, the monotonicity of $\{u_n\}$ implies uniform convergence of the entire sequence on $\{u_0 \geq 0\}$.

From weak convergence of the Monge-Ampère measure it follows that $|\omega_E(-u_0)| = 0$ for any Borel $E \subset \{u_0 > 0\} \setminus \overline{\Omega}$, hence u_0 is a generalized solution to $\det D^2 u = 0$ in $\{u > 0\} \setminus \overline{\Omega}$. We also get that $u_0 = h_0$ on $\partial\Omega$ as a result of the construction.

We next verify the free boundary condition for u_0. To accomplish this, fix any $z_0 \in \Gamma_{u_0}$ where the $(d-1)$-dimensional convex set Γ_{u_0} has a tangent hyperplane. We only need to check the condition for such points on the boundary. There is a sequence $z_n \in \Gamma_{u_n}$ converging to z_0. In view of concavity of u_n we may assume without loss of generality that each z_n is a regular point for Γ_{u_n}, and hence the graph of u_n will have a unique support hyperplane with slope λ_0; let H_n be this hyperplane. Up to passing to a subsequence, we may assume that unit normals of H_n converge to some unit vector $\nu_0 \in \mathbb{S}^d$. Let H_0 be the hyperplane of $\mathbb{R}^{d+1}$ through z_0 and with unit normal ν_0. By construction we get that H_0 is the unique support hyperplane to the graph of u_0 at z_0. Since all hyperplanes H_n are tangent to the graph of u_n and have slope λ_0, we obtain that H_0 is tangent to the graph of u_0 at z_0 and has slope λ_0, and hence the normal derivative of u_0 at z_0 equals λ_0, which is the free boundary condition at z_0.

Finally, to complete the proof, it is left to show that $u_0 = h_*$ on the set $\omega := \{h_* > 0\} \setminus \Omega$. To see this, let $\{X_n\}_{n=1}^{\infty} \subset \partial\widehat{\Omega}$ be the dense set of points fixed above, for which we have $u_0 = \inf_n H_{X_n}$. Then, clearly we get $h_*(x) \leq u_0(x)$ for all $x \in \omega$. Now if for some $y_0 \in \mathbb{R}^d \setminus \Omega$ we get $h_*(y_0) = 0$, then there is a sequence of support hyperplanes H_{Y_n} such that $H_{Y_n}(y_0) \to c_0 \leq 0$. Due to the density of X_n, up to passing to a subsequence, we may assume that $|X_n - Y_n| \to 0$ hence $|H_{X_n}(y_0) - H_{Y_n}(y_0)| \to 0$, which follows from the fact that at each point of $\partial\widehat{\Omega}$ the support hyperplane with slope λ_0 is unique by C^1-smoothness of $\partial\Omega$. We thus get $u_0(y_0) = 0$ and conclude that $h_* = u_0$ on ω. The proof of the proposition is complete. □

Remark 2.6. *For Ω without C^1 smoothness assumption, the proof of the previous proposition gives the existence of a solution to* (1.1)*, however we cannot claim that the solution obtained from polyhedral approximation must coincide with h_*.*

Our next result will be used to establish uniqueness of solutions to (1.1).

Lemma 2.7. (Segments in the graph) *Let u be any weak solution to* (1.1) *which is concave. Then, for any point $X^{(0)} = (x^0, u(x^0)) \in \mathbb{R}^{d+1}$ on the graph of u_0, there exists a line segment through $X^{(0)}$ lying entirely on the graph of u and having endpoints at the free boundary and on $\partial\widehat{\Omega}$.*

Proof. Let $\mathcal{M}$ be the graph of u. We will first prove the lemma when $X^{(0)}$ is in the interior of $\mathcal{M}$, i.e. $0 < u(x^0) < h_0$. In this case, let Π be any support hyperplane of $\mathcal{M}$ at $X^{(0)}$, and consider the convex hull $\mathcal{X} = \mathrm{Hull}(\Pi \cap \mathcal{M})$. Since Π is a support plane and u is concave, we get that $\mathcal{X}$ is a compact and convex set lying on the graph of u. To prove the lemma it is enough to see that $\mathcal{X}$ intersects the h_0- and 0-level surfaces of u, since then the statement of this lemma would follow by concavity of u. Now assume for contradiction that $\mathcal{X}$ stays on a positive distance from 0-level surface of u, or equivalently from $\mathbb{R}^d \times \{0\}$. We enclose the convex body bounded by $\mathcal{M}$ and hyperplanes $x_{d+1} = 0$ and $x_{d+1} = h_0$ by a convex polytope having one of its facets lying on the support plane Π. Moreover, we choose this polytope so that its boundary intersects $\mathcal{M}$ by $\mathcal{X}$ only. For $\varepsilon > 0$ small, applying [8, Theorem 1.1] due to M. Ghomi gives a convex body D_ε with C^∞ boundary such that $\partial D_\varepsilon \cap \mathcal{M} = \mathcal{X}$, $\partial D_\varepsilon \setminus \mathcal{X}$ has positive curvature, and the Hausdorff distance of D and the fixed polytope does not exceed ε. By choosing $\varepsilon > 0$ small enough and relying on the fact that Π is a support plane we can ensure that ∂D_ε stays in between Π and $\mathcal{M}$. Consequently, in the neighbourhood of $X^{(0)}$ the gradient image of the hypersurface ∂D_ε will be contained in the gradient image of $\mathcal{M}$. But this is a contradiction, since the latter has measure zero due to the equation, while the former has positive measure, in view of the strict convexity of ∂D_ε outside $\mathcal{X}$. This contradiction proves the lemma when $X^{(0)}$ is in the interior of the graph.

To complete the proof we are left to cover the case when $X^{(0)}$ is on the free boundary or on $\partial\widehat{\Omega}$. Assume the former, and fix a sequence of points $X^{(n)}$, $n \in \mathbb{N}$, in the interior of $\mathcal{M}$ converging to $X^{(0)}$. As we have proved already, for each n there is a support plane Π_n through $X^{(n)}$ intersecting the 0-level surface at some A_n and the 1-level surface at some B_n. Extracting a convergent subsequence, we may assume that the normal vectors ν_n of Π_n converge to some ν_0, and also that $A_n \to A_0$, $B_n \to B_0$ where $u(A_0) = 0$ and $u(B_0) = h_0$. By compactness we get that the $A_0 = X_0$ and the hyperplane through $X^{(0)}$ and having ν_0 as its normal is a support plane of $\mathcal{M}$. Again, due to compactness we get that $B_0 \in \Pi_0$, hence by concavity of u we have that the line segment $[X^{(0)}, B_0]$ lies on the graph of u.

The proof of the lemma is complete. □

We next establish a comparison principle with solutions having C^1 free boundaries or C^1 graphs.

Lemma 2.8. (Comparison principle) Let $\Omega_1 \subset \Omega_2$ be two convex domains, and let the concave function u_i be a weak solution to (1.1) corresponding to Ω_i, $i = 1, 2$. Define ω_i to be the closure of the convex hull of $\Gamma_i := \Gamma_{u_i}$, for $i = 1, 2$. We have the following:

(a) if either of Γ_i is C^1, then $\omega_1 \subset \omega_2$,

(b) if either of u_i is C^1 regular in $\{u_i > 0\} \setminus \overline{\Omega}$, then one has $U_1 \leq U_2$ on $\mathbb{R}^d$ where U_i stands for the extension of u_i into Ω_i as identically h_0.

Proof. We start with part (a). It is enough to prove the claim assuming that Γ_1 is C^1, since the other case follows a similar analysis. Assume for contradiction that ω_1 is not inside ω_2. Then, there is a point $X_2 \in \Gamma_2$ such that $u_1(X_2) > 0$, i.e. X_2 is in the interior of ω_1. Since Γ_2 is convex its normal exists almost everywhere, and without loss of generality we will assume that at X_2 the set Γ_2 has a unit outward normal, call it ν_2. Thanks to its smoothness at X_2 the set Γ_2 has a unique support hyperplane at X_2, call it $G_2 \subset \mathbb{R}^d \times \{0\}$ and hence, the graph of u_2 has a unique support plane with slope λ_0 at X_2; which we denote by $H_2 \subset \mathbb{R}^{d+1}$.

Since X_2 is in the interior of ω_1 translating G_2 and H_2 in the direction of ν_2 we will get a point $X_1 \in \Gamma_1$ where the shifted copies of G_2 and H_2, denoted respectively by G_1 and H_1, form a support hyperplanes to Γ_1 and the graph of u_1. Due to C^1 smoothness of Γ_1 we get that G_1 and H_1 are the only hyperplanes with those properties at X_1. Now recall that H_2 supports the graph of u_2, hence it has $\widehat{\Omega}_1$ on one side of it. But then, since H_1 is parallel to H_2 and is on the halfspace determined by H_1 and not containing Ω_1, we get that H_1 cannot intersect $\widehat{\Omega}_1$. The latter contradicts the statement of Lemma 2.7 as there should be a segment in the intersection of H_1 with the graph of u_1 joining the free boundary with the boundary of $\widehat{\Omega}_1$. The contradiction completes the proof of part (a).

The proof of part (b) follows from a similar idea, where we use support hyperplanes of the graph, and rely on the C^1-smoothness of one of the solutions. Thus, the proof of the lemma is complete. □

2.4. **Proof of Theorem A.** We start with the uniqueness of solutions.

Lemma 2.9. *Assume Ω is a bounded convex domain with $C^{1,1}$ boundary. Then* (1.1) *has a unique weak solution.*

Proof. Lemma 2.2 (see also Proposition 2.5) establishes the existence of a weak solution to (1.1) for $K_0 = 0$, and we only need to establish the uniqueness. Let u be any weak solution to (1.1) which is concave. By Lemma 2.2 we get a $C^{1,1}$-solution with $C^{1,1}$ regular free boundary. Hence, the comparison principle of Lemma 2.8 applied (twice) implies that u must coincide with this $C^{1,1}$ regular solution hence the proof. □

Remark 2.10. *In particular, we get that the envelope h_* given by* (2.8) *is $C^{1,1}$ as it coincides with the $C^{1,1}$ solution given by Lemma* 2.2.

Proof of Theorem A. The existence and uniqueness, as well as $C^{1,1}$ smoothness of the weak solution and the free boundary are given by Lemma 2.9. The definition of this solution in Lemma 2.2 shows that it is a ruled surface. The strict convexity of the free boundary, given that Ω is strictly convex, is due to construction of Lemma 2.2. The proof of the theorem is complete. □

3. Strictly elliptic case

Here we study the problem (1.1) with $K_0 > 0$, and prove Theorem B. The positivity of K_0 covers in particular the cases of positive curvature measure, and positive Gauss curvature. One crucial difference, however, from the homogeneous case, is that here (1.1) may have no solutions for arbitrary values of parameters involved in (1.1) as we show in the Appendix to this paper.

Throughout the section Ω is assumed to be C^2.

3.1. **Construction of a super-solution and Perron's method.** In this subsection we construct a solution to (1.1) via Perron's method. The set of all super-solutions to (1.1) in a sense of Definition 1.3 will be denoted by $\mathbb{W}_+(K_0, \lambda_0, \Omega)$. We will prove the existence of a solution to (1.1) by showing that the infimum over all super-solutions solves (1.1). As we show in the Appendix, there are arrangements of the parameters involved in the formulation of the problem (1.1), for which no solution exists. We thus start by formulating a technical condition under which the existence result will be established.

Let $\kappa_0 > 0$ be the minimum over all $x \in \partial\Omega$ of the smallest principal curvature of $\partial\Omega$ at x, and consider the inequality

$$K_0^{\frac{1}{d}} \leq \psi^{-\frac{1}{d}}(\lambda_0) \frac{\kappa_0 \lambda_0^2}{h_0 \kappa_0 + (\lambda_0^2 + h_0^2 \kappa_0^2)^{1/2}}. \tag{3.1}$$

Lemma 3.1. *Assume the function $\psi = \psi(\xi)$ in (1.1) is non-decreasing and smooth. Then, provided (3.1) holds, the set of super-solutions $\mathbb{W}_+$ is non-empty.*

Proof. We will construct a super-solution as a lower envelope of a certain family of paraboloids. Set $r_0 = 1/\kappa_0$, then for a given $x_0 \in \partial\Omega$, there is $z_0 \in \mathbb{R}^d$ such that the ball $B = B(z_0, r_0)$ is internally tangent to Ω at x_0 (i.e. B and Ω are tangent at x_0 and share the same outward normal at x_0). Moreover, for any $x \in \partial\Omega$ and any $x' \in \partial B$ satisfying $n_\Omega(x) = n_B(x')$ for the interior normal vectors, we have $II_{x,\Omega} \geq II_{x',B}$ for the second fundamental forms in view of the choice of r_0. Applying the inclusion principle for ovaloids due to Rauch [18], which is a generalization of Blaschke's rolling ball theorem, we conclude that $\Omega \subset B$.

Consider a paraboloid $P(x) = h_0 + \alpha r_0^2 - \alpha|x - z_0|^2$ where $\alpha > 0$ is a constant to be determined below, and r_0, z_0 are as above. It is clear that $P(x_0) = h_0$ and $P \geq h_0$ everywhere on $\overline{\Omega}$ due to the inclusion $\Omega \subset B$. We now choose α so that P will satisfy the first and the third conditions of Definition 1.3. A straightforward computation shows that the free boundary condition for P is equivalent to

$$2\alpha\left(\frac{h_0 + \alpha r_0^2}{\alpha}\right)^{\frac{1}{2}} = \lambda_0 \quad \Rightarrow \quad (4r_0^2)\alpha^2 + 4h_0\alpha - \lambda_0^2 = 0. \tag{3.2}$$

Solving the quadratic equation for positive α, and replacing $r_0 = 1/\kappa_0$ we get

$$\alpha = \frac{\lambda_0^2\kappa_0}{2\left(h_0\kappa_0 + (\lambda_0^2 + h_0^2\kappa_0^2)^{1/2}\right)}. \tag{3.3}$$

Using the fact that ψ is non-decreasing, we observe that P satisfies the inequality in Definition 1.3 (a) if

$$(2\alpha)^d \geq K_0\psi\left(2\alpha\left(\frac{h_0 + \alpha r_0^2}{\alpha}\right)^{1/2}\right) = K_0\psi(\lambda_0),$$

where we have also used (3.2). The obtained condition on α is precisely the inequality (3.1). The conclusion is that a single paraboloid satisfies a definition of a super-solution except for the boundary data. We next fix the boundary data, by taking the minimum over all paraboloids. To this end fix a dense sequence of points $\{x_n\}_{n=1}^\infty \subset \partial\Omega$ and for each $n \in \mathbb{N}$ let P_n be the paraboloid constructed above for x_n. For $n = 1, 2, ...$ define

$$u_n(x) = \min_{1\leq k\leq n} P_k(x), \qquad x \in \mathbb{R}^d.$$

Clearly, each u_n is concave as a minimum of concave functions. It is also clear, due to the construction of paraboloids, that $u_n(x_i) = h_0$ for each $1 \leq i \leq n$, and that $u_n(x) \geq h_0$ for all $x \in \overline{\Omega}$. We claim that $\det D^2(-u_n) \geq K_0\psi(|\nabla u_n|)$ on $\mathfrak{U}_n := \{u_n > 0\} \setminus \overline{\Omega}$. Indeed, assume $n = 2$, and fix any Borel set $E \subset \mathfrak{U}_2$. We partition E into the following subsets: $E_1 = \{x \in E : \ P_1(x) < P_2(x)\}$, $E_2 = \{x \in E : \ P_2(x) < P_1(x)\}$, and $E_{12} = \{x \in E : \ P_1(x) = P_2(x)\}$. In view of continuity of P_1, P_2 the sets E_1, E_2, E_{12} are Borel. For a given concave function f and Borel set B, let $\omega_B(f)$ be the gradient image of f on E. Then, we get

$$\omega_E(-u_2) = \omega_{E_1}(-u_2) \cup \omega_{E_2}(-u_2) \cup \omega_{E_{12}}(-u_2). \tag{3.4}$$

Using the theorem of A. D. Aleksandrov (see subsection 1.2.2), coupled with the fact that the sets E_1, E_2, E_{12} are pairwise disjoint, we obtain that the sets in the right-hand side of (3.4) intersect in null-sets. Also note that $\omega_{E_1}(-P_i) \subset \omega_{E_1}(-u_2)$ and $\omega_{E_{12}}(-P_i) \subset \omega_{E_{12}}(-u_2)$ for $i = 1, 2$. Hence we have

$$\begin{aligned}(3.5)\qquad \int_{\omega_E(-u_2)} \frac{d\xi}{\psi(|\xi|)} &= \int_{\omega_{E_1}(-u_2)} + \int_{\omega_{E_2}(-u_2)} + \int_{\omega_{E_{12}}(-u_2)} \geq \\ &\int_{\omega_{E_1}(-P_1)} + \int_{\omega_{E_2}(-P_2)} + \int_{\omega_{E_{12}}(-P_2)} \geq \\ &\qquad K_0|E_1| + K_0|E_2| + K_0|E_{12}| = K_0|E|.\end{aligned}$$

This shows that u_2 satisfies Definition 1.3 on $\mathfrak{U}_2$; the case for general $n \geq 2$ follows from the same argument by induction. We conclude that

$$(3.6)\qquad \det D^2(-u_n) \geq K_0\psi(|\nabla u_n|) \text{ on } \{u_n > 0\} \setminus \overline{\Omega}, \qquad \text{for each } n \in \mathbb{N}.$$

Due to the construction, for each $n \in \mathbb{N}$ we also have

$$(3.7)\qquad |\nabla u_n(x)| = \lambda_0, \qquad x \in \partial\{u_n > 0\},$$

in a weak sense, see subsection 1.2.1.

Consider the set $\mathfrak{U}_0 := \bigcap\limits_{n=1}^{\infty} \{u_n \geq -1\}$. In view of the construction of u_n, we have that $\mathfrak{U}_0$ is a compact convex set containing Ω. The sequence $\{u_n\}$ is a decreasing sequence of concave functions bounded below on $\mathfrak{U}_0$, and hence the limit $u_0(x) := \lim\limits_{n\to\infty} u_n(x)$, $x \in \mathfrak{U}_0$ exists and defines a concave function on $\mathfrak{U}_0$. It is clear from the density of the points $\{x_n\}$ in $\partial\Omega$ that $u_0 \equiv h_0$ on $\partial\Omega$, and also that $u_0 \geq h_0$ on Ω. A similar argument as in Proposition 2.5 shows that the convergence of u_n to u_0 is uniform on the set $\overline{\{u_0 > 0\}}$. In particular, from the weak continuity of the generalized Monge-Ampère measure (see [3, Theorem 9.1]) and (3.6) we get that $\det D^2(-u_0) \geq K_0\psi(|\nabla u_0|)$ on $\{u_0 > 0\} \setminus \overline{\Omega}$. Next, the free boundary condition for u_0 as claimed in Definition 1.3 (c) follows from the same line of argument as we had in Proposition 2.5 coupled with (3.7). Finally, redefining u_0 as identically equal to the constant h_0 on Ω, provides an element of $\mathbb{W}_+$, and completes the proof of this lemma. □

Remark 3.2. *Instead of paraboloids, one can construct a super-solution as an envelope of spherical caps, via a similar procedure. While this seems to be a natural choice for the equation of prescribed Gauss curvature, spherical caps need more restrictive assumptions, as compared to (3.1), for adjusting the free boundary condition.*

The next lemma is necessary for establishing a non-degeneracy result for super-solutions.

Lemma 3.3. (Comparison with vanishing curvature case) *Let u be any weak super-solution to (1.1), and let u_0 be the unique weak solution to (1.1) with identically 0 r.h.s., and with the same Ω, h_0 and λ_0. Then $u \geq u_0$ on $\{u_0 \geq 0\} \setminus \Omega$.*

Proof. Since Ω is assumed to be C^2 we have from Section 2 that u_0 is $C^{1,1}$ in its positivity set, and the free boundary $\partial\{u_0>0\}\setminus\overline{\Omega}$ is $C^{1,1}$ also. Now, the proof of the lemma follows from precisely the same argument as we had for Lemma 2.8. $\square$

In view of Lemma 3.1, the set of super-solutions $\mathbb{W}_+$ is non-empty and thanks to Lemma 3.3 all super-solutions are bounded below by the solution of the homogeneous equation with the same input data; in particular, super-solutions do not degenerate (collapse). Due to these observations, the lower envelope of $\mathbb{W}_+$

$$u_*(x):=\inf_{w\in\mathbb{W}_+} w(x), \qquad x\in\mathbb{R}^d,$$

defines a concave function bounded below by u_0-the solution to the homogeneous equation, and bounded above by the constant h_0. The aim is to show that u_* is a solution to (1.1). This we will do in two steps, first showing that u_* solves the equation in (1.1), and second, which is the hardest part, that u_* satisfies the free boundary condition.

Lemma 3.4. *Let u_* be as above. Then,*

1° *$u_*\in\mathbb{W}_+$ and*

$$\det D^2(-u_*)=K_0\psi(|\nabla u_*|) \quad \text{in} \quad \{u_*>0\}\setminus\overline{\Omega},$$

2° *the graph of u_* is strictly concave and consequently $u_*\in C^\infty$ in $\{u>0\}_*\setminus\overline{\Omega}$.*

Proof. **1°** We start by showing the existence of a minimizing sequence from $\mathbb{W}_+$ converging to u_*. Consider the set $U:=\{u_*\geq -1\}$ and $U_0:=\{u_*\geq 0\}\Subset U$, and fix any $\varepsilon>0$. All functions $w\in\mathbb{W}_+$ have uniform modulus of continuity in the set U_0 in view of [3, Lemma 3.1]. Hence, we can fix $\delta=\delta(\varepsilon)>0$ such that for all $w\in\mathbb{W}_+\cup\{u_*\}$ we get

$$|w(x)-w(y)|\leq\varepsilon, \text{ if } x,y\in U_0 \text{ and } |x-y|\leq\delta. \tag{3.8}$$

Next, we fix a set of points $\{x_i: \ 1\leq i\leq n\}\subset U_0$ forming a δ-net, i.e. for any $x\in U_0$ there is x_i with $|x-x_i|\leq\delta$. Now for each $1\leq i\leq n$ we fix $w_i\in\mathbb{W}_+$ such that $0\leq w_i(x_i)-u_*(x_i)\leq\varepsilon$. Since $w_\varepsilon:=\min_{1\leq i\leq n} w_i(x)\in\mathbb{W}_+$ we then get $0\leq w_\varepsilon(x_i)-u_*(x_i)\leq\varepsilon$ for all $1\leq i\leq n$. This, together with (3.8) easily implies that $0\leq w_\varepsilon(x)-u_*(x)\leq C\varepsilon$ for all $x\in U_0$ where $C>0$ is an absolute constant, and hence the existence of a minimizing sequence.

Using the existence of a minimizing sequence and weak continuity of Monge-Ampère measure, as we did in the proof of Lemma 3.1, we get $u_*\in\mathbb{W}_+$, and to complete the proof of the current lemma, it is left to show that u_* solves the PDE.

We have $\det D^2(-u_*)\geq K_0\psi(|\nabla u_*|)$ on $\{u_*>0\}\setminus\overline{\Omega}$, and assume, for contradiction, that there is a ball $B\subset\{u_*>0\}$ such that $\int_{\omega_B(-u_*)}\frac{d\xi}{\psi(|\xi|)}>K_0|B|$. Let u be the concave solution to $\det D^2(-u)=K_0\psi(|\nabla u|)$ in B and $u=u_*$ on ∂B. The existence of the solution u to Dirichlet's problem in strictly convex domain for continuous boundary data is well-known (see [3, Theorem 10.1]). Now consider the function $\widetilde{u}$ equal to u on B and u_* otherwise. By construction $\widetilde{u}$ is concave, and following the same lines as in (3.5) we have that $\widetilde{u}\in\mathbb{W}_+$. Due to the comparison principle, we get $u < u_*$ in B, hence $\widetilde{u}$ is smaller than the infimum u_* in B which gives a contradiction and completes the proof of part **1**°.

2° Since u_* solves the PDE by **1**°, to show its smoothness it is enough to prove that u_* is strictly concave (see [17], and [6]). To this end, we use an argument from [22, section 5.6]. If Graph_{u_*} is not strictly concave then there is a line segment $\ell \subset \mathrm{Graph}_{u_*}$. Since u_* solves the PDE in view of part **1**°, there are two positive constants μ_1, μ_2 such that $\mu_1 \le \det D^2(-u_*) \le \mu_2$. Then, it follows from [4] that ℓ cannot have extremal points in the relative interior of Graph_{u_*}. Hence there is $z_1 \in \partial\widehat{\Omega}$ such that $z_1 \in \partial\widehat{\Omega} \cap \ell$. Choosing a new coordinate system we can assume that $z_1 = 0$ and ℓ is the x_d axis. Observe that ℓ is transversal to $\partial\widehat{\Omega}$. From the regularity of Ω it follows that for the convex function $\widetilde{u}$ representing the surface in the new coordinate system

$$0 \le \widetilde{u}(x_1, \dots, x_{d-1}, x_d) \le C \sum_{i=1}^{d-1} x_i^2, \tag{3.9}$$

near the origin for $x_d > 0$ sufficiently small. However, using the positivity of the right hand side of the PDE, one can construct a sub-solution w near the origin, in a small ball $B := B(\varepsilon e_d, \varepsilon)$, $\varepsilon > 0$, say, with $w \in C^2(\overline{B})$ and $\widetilde{u} \ge w \ge 0$ in view of the comparison principle. But the latter violates (3.9), as $\partial^2_{dd} w(0) > 0$ thanks to the smoothness of w and condition $\det D^2 w > 0$ in B. This completes the proof of part **2**°, and hence of the lemma. ☐

3.2. **Blaschke extension of the minimal super-solution.**

The aim of this subsection is to show that the smallest super-solution u_* verifies the free boundary gradient condition. To this end, we will extend u_* in a neighbourhood of each free boundary point below the halfspace $\mathbb{R}^d \times \{0\}$ as a super-solution to our problem. This extension will reduce the matters to a similar scenario as we had in the interior case. The construction of the extension of u_* is inspired by Blaschke's rolling ball theorem, and hence the name of the extension.

In the construction below all support hyperplanes of u_* at the free boundary points are **extreme**, that is the hyperplanes cannot be further tilted towards Graph_{u_*} (see subsection 1.2.1).

Take any point $x \in \Gamma_{u_*}$, if it is a regular point of the free boundary, then there is a unique support hyperplane in $\mathbb{R}^{d+1}$ to the graph of u_* passing through $X = (x, 0) \in \mathbb{R}^{d+1}$ and having slope λ_0. We call this plane H_x. Otherwise, if x is not a regular point, we proceed with any support hyperplane to Graph_{u_*} through X having slope[3] λ_0. Next, let $H_x^\perp$ be the hyperplane passing through $H_x \cap (\mathbb{R}^d \times \{0\})$ and the normal to H_x. Clearly, $H_x^\perp$ is orthogonal to H_x. We now define the convex body $\mathcal{S}_*^+$ bounded by Graph_{u_*} if $0 < x_{d+1} < h_0$, $\Omega \times \{h_0\}$ if $x_{d+1} = h_0$, and when $x_{d+1} < 0$ we define $\mathcal{S}_*^+$ as the intersection of all half spaces corresponding to the extreme support functions at Γ_{u_*}, which contain Ω.

---

[3] Taking a sequence $x_d \in \Gamma_{u_*}$ of regular points converging to x and defining H_x as the limit of the corresponding hyperplanes H_{x_n}, produces a support hyperplane with slope λ_0 at a non-regular point x.

For the given $x \in \Gamma_{u_*}$ consider $\mathcal{S}_x^-$ as the mirror reflection of $\mathcal{S}_*^+$ with respect to $H_x^\perp$ (see Figure 3). Let $x_1, \dots x_k$ be distinct points on Γ_{u*} near fixed $x_0 \in \Gamma_{u*}$ and let

$$\mathcal{S}^m = \mathcal{S}_*^+ \cap \bigcap_{j=1}^m \mathcal{S}_{x_j}^-.$$

Note that $x_j \in \mathcal{S}^m$ for every $j = 1, \dots, m$ and $\mathcal{S}^m$ is a bounded convex body.

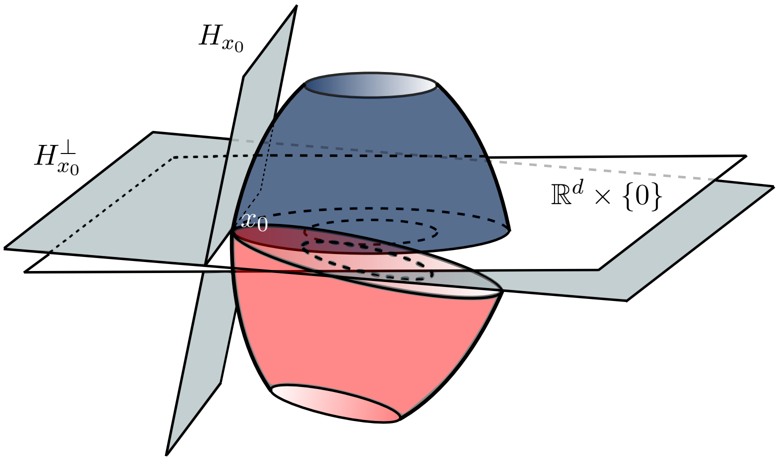

FIGURE 3. The light red coloured surface is the reflection of the original surface (coloured in blue) with respect to the hyperplane $H_{x_0}^\perp$, where x_0 is a fixed point on the free boundary. To construct the Blaschke extension, we first extend u_* for $x_{d+1} < 0$ by the lower envelope of its extreme supporting planes at the free boundary points, and then do the reflection described in the figure, for the extended u_*.

The projections of the free boundaries of the reflections onto $T_0 = \mathbb{R}^d \times \{0\}$ always contain Γ_u near the contact point. Thus we can choose a dense sequence of points near a fixed point x_0 on the free boundary and let $m \to \infty$ to obtain a convex body $\mathcal{S}_B$ as the limit of the nested sequence of $\mathcal{S}^m$. It follows that $\Gamma_{u_*} \subset \partial \mathcal{S}_B$ in a neighbourhood of x_0. We call $\mathcal{S}_B$ the **Blaschke reflection body**. Clearly, $\mathcal{S}_B$ defined in this way, depends on x_0 and the fixed neighbourhood of it. However, we will always work locally in a neighbourhood of a given point, and for this reason there is no need to keep track of the dependence of $\mathcal{S}_B$ on a point x_0.

Using $\mathcal{S}_B$ as a barrier we wish to show that u_* satisfies the free boundary condition in generalized sense (subsection 1.2.1). To begin with, we first look at the regular free boundary points where the free boundary condition may fail.

Lemma 3.5. *Let $x_0 \in \Gamma_{u_*}$ be a regular point of the free boundary, and such that the slope at x_0 satisfies*

$$|\nabla u_*(x_0)| < \lambda_0.$$

Then the boundary of the convex body $\mathcal{S}_B$ near x_0 is a graph in the direction of x_{d+1} axis and has slope strictly less than λ_0. Here $\mathcal{S}_B$ is the Blaschke reflection body which contains $\mathrm{Hull}(\mathrm{Graph}_{u_*}) = \mathcal{S}_*^+ \cap \{x_{d+1} \geq 0\}$.

Proof. Suppose the claim fails, then there are points $p_k \in \mathcal{S}_B \cap \{x_{d+1} < 0\}$ with support functions at p_k such that slopes $> \lambda_0$.

Recall that $\mathcal{S}_B = \mathcal{S}_*^+ \cap \bigcap_{x \in \Gamma_{u_*}} S_x^-$ therefore there must be $x_k \in \Gamma_{u_*}$ such that $p_k \in \mathcal{S}_{x_k}^-$. Denote $M_k \in \text{Graph}_{u_*}$ the pull back of p_k. By construction $M_k \notin \Gamma_{u_*}$ because there the slopes are $\le \lambda_0$. Thus at each $M_k \in \text{Graph}_{u_*}$ there is a unique support hyperplane H_k with slope $> \lambda_0$. Since $M_k \to x_0$ and x_0 is a differentiability point of Γ_{u_*} then we get the limit of H_k is an extreme (as discussed in Remark 1.1) support hyperplane of Graph_{u_*} at x_0 with slope λ_0 (see [3, Lemma 9.1] for convergence) but this is impossible due to our assumption. Now the fact that $\mathcal{S}_B$ is a graph near x_0 easily follows from the first claim that the slope at x_0 is $< \lambda_0$. $\square$

Corollary 3.6. *For each point $p = (p_1, \dots, p_{d+1})$ on $\mathcal{S}_B$ with $p_{d+1} < 0$ we have that near x_0 the boundary of $\mathcal{S}_B$ is a graph of some concave function U_* over $\mathbb{R}^d \times \{0\}$ and* $\det D^2(-U_*) \ge K_0 \psi(|\nabla U_*|)$.

Proof. By Lemma 3.5, the boundary of $\mathcal{S}_B$ is a graph near x_0 over $\mathbb{R}^d \times \{0\}$. Then apply the argument in the proof of (3.6) to complete the proof of this corollary. $\square$

From Corollary 3.6 and Lemma 3.5 it follows that there must be a neighbourhood $\mathcal{N}$ of x_0 in $\mathbb{R}^{d+1}$ such that on $(\mathcal{N} \cap \mathcal{S}_B) \cap \{x_{d+1} < 0\}$ the slopes are $< \lambda_0$. If not then we will get a ruled piece of $\mathcal{S}_B$ in $\{x_{d+1} < 0\}$ where the curvature must vanish.

With these preparations, we are now ready to show that u_* satisfies the free boundary condition.

Lemma 3.7. *For all regular points $x_0 \in \Gamma_{u_*}$ we have $|\nabla u_*(x_0)| = \lambda_0$.*

Proof. Assume for contradiction, that $x_0 \in \Gamma_{u_*}$ is a regular point where $|\nabla u_*(x_0)| < \lambda_0$. Let H_0 be the hyperplane in $\mathbb{R}^{d+1}$ supporting the graph of u_* at $(x_0, 0)$ and having slope λ_0. Notice that H_0 is the tilted copy of the support hyperplane H_{x_0}. By construction of the extension $\mathcal{S}_B$, we have that H_0 cuts a non-trivial cap from $\mathcal{S}_B$.

We next take $\delta > 0$ small and let H_δ be the parallel translation of H_0 by $\delta > 0$ towards Ω. For $\delta > 0$ small enough we get that $\mathcal{S}_B$ is seen as a concave function $\widehat{U}_*$ from H_δ. Since the equation is invariant with respect to rotations and translations in T_0, without loss of generality we assume that $x_0 = 0$ and the inner normal at x_0 is in the direction of x_1 axis. Then H_δ can be represented as $x_{d+1} = \lambda_0(x_1 - \delta)$. For sufficiently small δ the set $\mathscr{D}_\delta = \{x \in T_0 : U_* > \lambda_0(x_1 - \delta)\}$ is a convex domain that can be projected on T_0 in one-to-one fashion thanks to Corollary 3.6. Let $\widehat{U}_* = u_* - \lambda_0(x_1 - \delta)$ and $\widehat{u}_\delta$ be the unique solution to

$$\begin{cases} \det D^2(-\widehat{u}_\delta) = K_0 \psi(|\nabla \widehat{u}_\delta + \lambda_0 e_1|) & \text{in } \mathscr{D}_\delta, \\ \widehat{u}_\delta = 0 & \text{on } \partial \mathscr{D}_\delta. \end{cases} \tag{3.10}$$

We claim that $\widehat{u}_\delta < \widehat{U}_*$. Indeed, to see this it is enough to show that

$$\det D^2(-\widehat{U}_*) > K_0 \psi(|\nabla \widehat{U}_* + \lambda_0 e_1|) \qquad \text{in } \{\widehat{U}_* > 0\}, \tag{3.11}$$

as then the desired inequality will follow from the comparison principle. Observe, that we have a non-strict inequality in (3.11) by construction (see e.g. the proof of

Lemma 3.1). For the strict inequality, we notice that for each X on the graph of U_* there exists (by compactness) a reflected function $U_{*,x}$ (representing $\mathcal{S}_x^-$) touching the graph of U_* at X having its pull back on $\mathcal{S}_B \cap \mathrm{Graph}_{u_*}$ as interior point (which is possible by construction) and staying above Graph_{U_*}. Since $\widehat{U}_{*,x} := U_{*,x} - \lambda_0(x_1-\delta)$ solves the PDE in (3.11) with equality, then Hopf's lemma (see Lemma 3.4) implies that the two functions $\widehat{U}_{*,x}$ and $\widehat{U}_*$ must coincide unless we have a strict inequality in (3.11).

We thus obtain that $\widehat{u}_\delta < \widehat{U}_*$, and to complete the argument it is left to show that $u_\delta = \widehat{u}_\delta + \lambda_0(x_1-\delta)$ intersects $\mathbb{R}^d \times \{0\} = \{x_{d+1} = 0\}$ by a slope bounded above by λ_0. This will then produce a smaller super-solution to (1.1), contradicting the definition of u_*. Assume that for any $\delta > 0$ small there exists a point $x_\delta \in \Gamma_{u_\delta}$ such that the slope of u_δ at x_δ is larger than λ_0. Extracting a sequence $\delta_j \to 0$ we get that $x_{\delta_j} \to x_0$, and $u_{\delta_j} \to U_*$ in the set $\{U_* \leq 0\}$, where U_* is defined (near x_0) so that $\mathrm{Graph}_{U_*} = \{-\varepsilon < x_{d+1}\} \cap \mathcal{S}_B$ for some small $\varepsilon > 0$ depending on x_0, see Lemma 3.6. But then we should have that the support plane of u_{δ_j} converges to the support plane of U_* at x_0 as $j \to \infty$. However, the slope of (any) limit of these planes has magnitude at least λ_0, which by construction cannot be a support plane of the Blaschke extension at x_0. This contradiction completes the proof of the gradient condition on the free boundary. □

3.3. C^∞ regularity of the free boundary.

In this section we complete the proof of Theorem B. We recapitulate the statement as follows

Proposition 3.8. *There is $K_0 > 0$ small enough depending only on λ_0, Ω, and ψ such that for any $0 < K < K_0$ the minimal solution u_* to (1.1) with curvature measure K has C^∞ smooth free boundary.*

The idea is to reduce the regularity of the free boundary to interior regularity by extending a solution from the free boundary below $\mathbb{R}^d \times \{0\}$. For this we first show that the free boundary is C^1 regular for small enough K, by reducing matters to the case of $K = 0$. Relying on C^1 regularity of the free boundary and interior smoothness of solutions we show that the free boundary rolls freely inside a ball of a uniform radius. This enables us to apply the same construction, as we used for establishing the existence of a solution, starting from the convex hull of the free boundary as the initial domain Ω.

Sublemma 3.9. *Assume that $\psi \in C^\infty$ such that $\psi > 0$ and $\sup_{B(0,\lambda_0)} \psi(\xi) < \infty$. Let u_K be the minimal solution of*

$$\begin{cases} \det D^2(-u_K) = K\psi(|\nabla u_K|), & \{u_K > 0\} \setminus \overline{\Omega}, \\ u_K = h_0, & \partial\Omega, \\ |\nabla u_K| = \lambda_0, & \Gamma_{u_K}. \end{cases} \tag{3.12}$$

corresponding to the given constant $K > 0$.

1° *There is a K_0 small such that for any $0 < K < K_0$ the normal of the free boundary Γ_{u_K} is uniformly continuous (consequently Γ_{u_K} are uniformly C^1).*

In other words, there is a critical K_0 small such that the following holds: for every $\varepsilon > 0$ there is $\delta > 0$ such that if

$$\sup_{\substack{x_K, y_K \in \Gamma_{u_K} \\ K \in (0, K_0)}} |x_K - y_K| < \delta$$

then

$$\sup_{\substack{x_K, y_K \in \Gamma_{u_K} \\ K \in (0, K_0)}} |\nu(x_K) - \nu(y_K)| < \varepsilon.$$

2° *Moreover $\nabla u_K|_{\Gamma_{u_k}} = \lambda_0 \nu$ where ν is the unit inner normal of Γ_{u_K}.*

Proof. **1°** If the claim fails for some $\varepsilon_0 > 0$ then there are sequences $K_j \downarrow 0$, solutions $u_j := u_{K_j}$ to (1.1), and points $x_j, y_j \in \Gamma_{u_j}$ such that

$$|x_j - y_j| < \frac{1}{j}, \quad \text{but} \quad |\nu(x_j) - \nu(y_j)| \geq \varepsilon_0 > 0.$$

Recall that u_j is the minimal solution defined by $u_j = \inf_{\mathbb{W}_j} v$, where

$$\mathbb{W}_j := \mathbb{W}_-(K_j, \lambda, \Omega) = \left\{ v \text{ concave}: \ \det(D^2(-v)) \geq K_j \psi(|\nabla v|), \quad |\nabla v(\Gamma_{u_K})| \leq \lambda_0 \right\}.$$

Note that (3.1) holds. Clearly for u_1 we have $\dfrac{\det(-D^2 u_1)}{\psi(|\nabla u_1|)} = K_1 \geq K_j$ therefore

$$u_1 \in \mathbb{W}_j.$$

Consequently $u_j \leq u_1$ and

$$\{u_j > 0\} \subset \{u_1 > 0\}, \quad \forall j \geq 1.$$

This provides uniform bound for the convex bodies of $\mathscr{K}_j = \text{Hull}(\text{Graph}_{u_j})$ and applying Blaschke's selection principle it follows that for some subsequence $\{j_m\}$ there is a concave function $u_0 = \lim_{j_m \to \infty} u_{j_m}$ such that $\det(-D^2 u_0) = 0$ in $\{u_0 > 0\} \setminus \overline{\Omega}$.

We claim that $\left|\nabla u_0|_{\Gamma_{u_0}}\right| = \lambda_0$. If not then there is a point $z_0 \in \mathbb{R}^d \setminus \text{Hull}(\Gamma_{u_0})$ such that u_0 at z_0 has an extreme support hyperplane H with slope $\lambda < \lambda_0$. Due to uniform convergence near $\partial\Omega$ we see that H must touch $\partial\Omega$ at some ξ_0. Consequently we can tilt H at ξ_0 and obtain another plane H_δ with larger slope $\lambda + \delta < \lambda_0$, again touching Ω at ξ_0. Choosing δ small enough and using the construction of (3.10) with respect to the hyperplane H_δ we can construct a solution $\widehat{u}_\delta$ such that $\sup \widehat{u}_\delta \leq C K_j$ for K_j sufficiently small (this follows from the Aleksandrov maximum principle). With δ small enough we can construct a super-solution from $\widehat{u}_\delta$ which is below u_{K_j} and consequently we will reach a contradiction since u_{K_j} is minimal.

Passing to the limit we have that at $x_0 = \lim_{j_m} x_{j_m}$ the support hyperplanes of codimension 1 at the free boundary of u_j will converge to two distinct support hyperplanes with normals $\nu_1(x_0), \nu_2(x_0)$ at x_0. We get

$$|\nu_1(x_0) - \nu_2(x_0)| \geq \varepsilon_0$$

which is a contradiction in view of Theorem A.

2° Let $x_i \subset \{u > 0\}$ such that $x_i \to x_0 \in \Gamma_u$. Let ν_0 be the outer unit normal at x_0, which is unique by the first part **1°**. We can extract a subsequence i_m such that $\{\nabla u(x_{i_m})\}$ is convergent to some vector ξ_0, and the gradient estimate forces

$|\xi_0| \le \lambda$. Since the tangent planes at x_i must converge to an extreme support plane of Graph_u at x_0, it follows that ξ_0 and ν_0 are collinear. Suppose that $|\xi_0| < \lambda_0$ then $x_{d+1} = \xi_0 \cdot (x - x_0)$ is an extreme support hyperplane at x_0 which is in contradiction with the fact that at x_0 the free boundary condition is satisfied in the classical sense. Hence $|\xi_0| = \lambda_0$ and the desired result follows. □

Lemma 3.10. *Let u be as in Proposition 3.8. Then there is a constant R_0 depending only on λ_0, d, ψ, and independent of K_0, such that for every $x_0 \in \Gamma_u$ there is z_0 such that $x_0 \in \partial B(z_0, R_0)$ and $\Gamma_u \subset B(z_0, R_0)$.*

Proof. Let the origin $0 \in \Omega$ and v be the Legendre transform of $-u$. Then we have that $\det D^2 v = \frac{1}{K_0\psi(|z|)}$ in $0 < |z| \le \lambda_0$ (cf. Lemma 3.4). On the boundary $|z| = \lambda_0$, in view of Sublemma 3.9, we have that $v(z) = x \cdot z = \lambda_0 \nu \cdot \nabla v$, ν is the unit inner normal. Thus v verifies the oblique boundary condition $\lambda_0 v_\nu + v = 0$ on $|z| = \lambda_0$. Note that v is C^∞ smooth near the sphere $|z| = \lambda_0 - \delta$, for sufficiently small $\delta > 0$. Taking the logarithm $\log\det D^2 v = -\log K_0 - \log\psi(|z|)$ and applying the $C^{1,1}$ boundary estimates [24, pp. 23-29] we get

$$\sup_{\partial B(0,\lambda_0)} |D^2 v| \le C$$

where $C = C(d, \lambda_0, \|\log\psi\|_{C^{1,1}})$ does not depend on K_0. Pulling back to $-u$ we infer that

$$\liminf_{x\to x_0} D^2(-u(x)) \ge c_0 \mathrm{Id}, \quad \forall x \in \{u > 0\}, \quad \forall x_0 \in \Gamma_u.$$

for some positive constant c_0 and the result follows. □

Consider the problem (1.1) with K_0 sufficiently small, such that there exists a solution u with free boundary rolling freely inside a ball of radius R_0. By Lemma 3.10 we have that smallness of K_0 is sufficient for this purpose. Set $\Omega_u = \mathrm{Hull}(\Gamma_u)$, we claim that in a neighbourhood of Ω_u contained in $\mathbb{R}^d \setminus \overline{\Omega_u}$ the minimal solution u can be extended as a solution.

More precisely, there exists $U : \mathbb{R}^d \setminus \Omega_u \to \mathbb{R}$ concave solving the following problem:

$$\begin{cases} \det D^2(-U) = K_0\psi(|\nabla U|), & (\mathbb{R}^d \setminus \overline{\Omega_u}), \\ U = 0, & \Gamma_u, \\ |\nabla U| = \lambda_0, & \Gamma_u. \end{cases} \tag{3.13}$$

To construct such extension, we will treat Ω_u as the new initial domain Ω. We next modify slightly the Definition 1.3, by defining the set $\mathbb{W}_- = \mathbb{W}_-(K_0, \lambda_0, \Omega_u)$ as the class of all concave functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\begin{cases} \det D^2(-u) \ge K_0\psi(|\nabla u|), & \{u < 0\} \setminus \overline{\Omega_u}, \\ u \equiv 0, & \overline{\Omega_u}, \\ |\nabla u| \le \lambda_0, & \partial\Omega_u. \end{cases} \tag{3.14}$$

Next, relying on Lemma 3.10, which ensures that Ω_u rolls freely inside a ball of some radius R_0, we invoke the Perron method utilized in Section 3.1 with the gradient condition on the free boundary replaced by gradient condition on the boundary of the initial domain. The details are the same, and we will omit them.

Finally, defining

$$\widetilde{u} = \begin{cases} u & \{u > 0\} \setminus \overline{\Omega}, \\ U, & \{U < 0\}, \end{cases} \tag{3.15}$$

and using the fact that the gradients of u and U agree on the free boundary Γ_u we easily get that $\widetilde{u}$ defines a solution across the free boundary Γ_u. Applying the local C^2 estimates of Pogorelov [17], [6] to $\widetilde{u}$ the result of Proposition 3.8 follows.

Proof of Theorem B. Putting together Lemmas 3.4 and Lemma 3.7 we get that u_* defines a solution to (1.1). The regularity of the free boundary $\Gamma_{u_*}$is given by Proposition 3.8. □

Appendix A. Non existence of K-surfaces

The aim of this appendix is to show, based on an example of radial solutions, that depending on the parameters of (1.1), there can be no solution to (1.1).

A.1. **Radially symmetric solutions.**

In this subsection, using the moving plane method of Aleksandrov, we show that any solution to (1.1) with $K_0 > 0$ and Ω a ball, must be radially symmetric. Relying on this, we next show, by an explicit example, that there might be no solution to (1.1) for general values of parameters.

Lemma A.1. *Let u be a uniformly convex solution of (1.1) such that $\psi(\xi) = \psi(|\xi|)$, $\xi \in \mathbb{R}^d$, Ω is a ball centred at the origin and let the free boundary Γ_u be C^2. Then u is radially symmetric and the free boundary is sphere of codimension 2.*

Proof. We will use Aleksandrov's moving plane method [21]. Let $e_1 \in \{x_{d+1} = 0\}$, be the unit direction of the x_1 axis and T a hyperplane perpendicular to e_1. In other words, T is defined as the set of all points $X \in \mathbb{R}^{d+1}$ such that $x_1 = t$ where $t \in \mathbb{R}$. This is a one parameter family of parallel planes moving orthogonal to e_1. Let us move the plane T from $t = -\infty$ until it first time hits Γ_u, say at $t = t_1$. Then for each $t > t_1$ we let

$$D_t = \bigcup_{s \in (t_1, t)} T_t \cap \{u > 0\}$$

and D_t^* be the symmetric domain of D_t with respect to T_t. Similarly, we let S_t to be the surface that T cuts from Graph_u and S_t^* its symmetric reflection with respect to T_t.

We consider few cases:

Case 1: Let $t_0 > t_1$ be such that S_t^* and Graph_u touch first time at some interior point $x_0 \in \{u > 0\} \setminus \overline{\Omega}$ then letting $u^*(x) = u(x^*)$ where x^* is the symmetric point determined from the equality

$$x_1^* = 2t - x_1, \quad x_i^* = x_i, \quad i = 2, \ldots, d.$$

If we denote $(x_1, x_2, \dots, x_d) = (x_1, y)$ then $u^*(x^*) = u(2t - x_1, y)$. Clearly,

$$\begin{aligned} \det D^2(-u^*(x)) &= (\det D^2(-u))(2t - x_1, y) \\ &= \psi(\nabla u^*) \\ &= \psi((\nabla u)(2t - x_1, y)) \end{aligned}$$

and consequently we have that

$$\begin{aligned} &\text{(A.1)} \quad \log\psi(\nabla u^*) - \log\psi(\nabla u) &&= \log\det D^2(-u^*(x)) - \log\det D^2(-u(x)) \\ &\text{(A.2)} &&= a_{ij}[u^*_{ij}(x) - u_{ij}(x)] \end{aligned}$$

where

$$\begin{aligned} a_{ij}(x) &= \int_0^1 \frac{\partial \det(-D^2(su^*(x) + (1-s)u(x)))}{\partial r_{ij}} ds \\ &= \int_0^1 \left[\operatorname{cof}\left(-D^2\left(su^*(x) + (1-s)u(x)\right)\right)_{ij}\right] ds. \end{aligned}$$

Here cof is the cofactor matrix. By assumption u is uniformly convex, i.e. $-D^2u(x) \geq c\mathrm{Id}, c > 0, x \in \{u > 0\} \setminus \overline{\Omega}$ thus a_{ij} is uniformly elliptic matrix with ellipticity constants depending only on c and $C^{2,\alpha}$ norm of u (note that if $\Gamma_u \in C^{2,\alpha}$ then from existing theory of K-surfaces we can conclude that $\mathrm{Graph}_u \in C^{2,\alpha}(\overline{\{u > 0\} \setminus \Omega})$). Similar argument implies that

$$\log\psi(\nabla u^*) - \log\psi(\nabla u) = b \cdot (\nabla u^* - \nabla u), \quad b = \int_0^1 \frac{\nabla\psi(s\nabla u^* + (1-s)\nabla u)}{\psi(s\nabla u^* + (1-s)\nabla u)} ds.$$

At the touching point we can apply Hopf's maximum principle to the linearised equation $a_{ij}w_{ij} - b_i w_i = 0, w = u^* - u$ to conclude that $u^* = u$ in D_t^*. Hence if we have interior touch between S_t^* and Graph_u then Graph_u must be symmetric w.r.t. T_{t_0}.

Case 2: Suppose that the first touch of S_t^* and Graph_u happens at the boundary $x_0 \in \Gamma_u \cap \partial D_t^*$. Since we are assuming that the free boundary is $C^{2,\alpha}$ smooth then at x_0 we have that $|\nabla u(x_0)| = |\nabla u^*(x_0)| = \lambda_0$. We can apply Hopf's lemma at x_0 to infer that Graph_u is symmetric w.r.t. T_{t_0}

Case 3: Finally let us consider the case when $x_0 \in T_{t_0} \cap \Gamma_u$, where t_0 is the value for which S_t^* and Graph_u touch the first time. For this case we need to apply a well-known computation by Serrin [21]. We consider an orthonormal frame at x_0, the x_n axis being directed along the inward normal to Γ_u. Assuming that near x_0 the free boundary has the representation

$$x_n = \phi(x_1, \dots, x_{n-1})$$

for a convex function $\phi \in C^{2,\alpha}$. Using Serrin's computation [21, pp. 307-308] we have

$$D^2u(x_0) = \left[\begin{array}{c|c} -\lambda\phi_{ij} & 0 \\ \hline 0 & u_{nn} \end{array}\right]$$

From the equation $\det(-D^2u) = K\psi(\nabla u)$ we find that

$$-u_{nn} = \frac{\psi(\lambda)}{\lambda^{n-1}\det\phi_{ij}}.$$

Hence at x_0 u and u^* have identical gradient and Hessian. Applying the Lemma 2 from [21] we again conclude that Graph_u is symmetric w.r.t. the plane T_{t_0}.

Summarizing, we see that in any case for each unit vector e there is a position for the plane T_t such that Graph_u is symmetric w.r.t. T_{t_0}. Suppose that T_{t_0} (the one where the first touch happens for given unit vector e) does not intersect Ω. From above analysis it follows that $S^*_{t_0}$ must contain horizontal hole $\Omega \times \{1\}$. But S_{t_0} and $S^*_{t_0}$ are symmetric hence both must contain the horizontal holes. Thus T_{t_0} must intersect $\Omega = B(0,1)$. Since Ω is symmetric w.r.t. 0 then it follows that T_{t_0} passes through the origin. Since e is arbitrary then it follows that u is radially symmetric. □

A.2. **Explicit computations.** Consider (1.1) with $\Omega = B(0, R_0)$ and $g \equiv h_0 > 0$ on the boundary of Ω. We search for a radial solution to (1.1) with given $K = K_0 > 0$, $\lambda = \lambda_0$ under these conditions. Let u be radial, i.e. there exists a function $v : [R_0, \infty) \to \mathbb{R}_+$ such that $u(x) = v(r)$ for $|x| = r$. Then, a direct computation reveals

$$u_{ii}(x) = \frac{x_i^2}{r^2}\left(\ddot v - \frac{\dot v}{r}\right) + \frac{\dot v}{r} \qquad \text{and} \qquad u_{ij}(x) = \frac{x_i x_j}{r^2}\left(\ddot v - \frac{\dot v}{r}\right), \tag{A.3}$$

where $1 \le i \ne j \le d$. To compute the determinant of the Hessian of u we will rely on *matrix-determinant lemma*, which states that for invertible $A \in M_d(\mathbb{R})$ and column-vectors $u, v \in \mathbb{R}^d$ one has $\det(A + uv^T) = (1 + v^T A^{-1}u)\det A$. To apply this identity, set $\ddot v - \frac{\dot v}{r} := \alpha$, $\frac{\dot v}{r} := \beta$, and $u^T := \frac{1}{r}(x_1, ..., x_d)$; we get

$$\det D^2 u = \det(\alpha u u^T + \beta \mathrm{Id}) := \mathcal{D}.$$

We will first assume that both α and β are non-zero, in which case we obtain

$$\mathcal{D} = \alpha^d \det\left(uu^T + \frac{\beta}{\alpha}\mathrm{Id}\right) = \text{(matrix-determinant lemma)} \tag{A.4}$$
$$\alpha^d\left(1 + u^T\frac{\alpha}{\beta}\mathrm{Id}\, u\right)\det\left(\frac{\beta}{\alpha}\mathrm{Id}\right) = \alpha^d\left(1 + \frac{\alpha}{\beta}|u|^2\right)\left(\frac{\beta}{\alpha}\right)^d = \beta^d + \alpha\beta^{d-1},$$

where the last equality is in view of the identity $|u| = 1$. We now observe that the formula (A.4) remains valid when either of α or β becomes 0, we thus have (A.4) for all α, β. We now plug the values of α, β into the formula for (A.4), and obtain $\det D^2 u = \ddot v\left(\frac{\dot v}{r}\right)^{d-1}$. Finally, getting back to (1.1), the equation becomes

$$\ddot v\left(\frac{\dot v}{r}\right)^{d-1} = (-1)^d K_0. \tag{A.5}$$

From this ODE we get

$$(\dot v(r))^d = (-1)^d K_0 r^d + A, \tag{A.6}$$

where A is some constant. One has two cases here.

Case 1. Take $A = 0$. Then $u(x) = B - K_0^{1/d}\frac{|x|^2}{2}$ for a constant B. From the boundary condition on u we get

$$u(x) = h_0 + \frac{K_0^{1/d}}{2}(R_0^2 - |x|^2). \tag{A.7}$$

Finally, the free boundary condition is satisfied, if and only if

$$\lambda_0 = K_0^{1/d}\left(R_0^2 + 2h_0 K_0^{-1/d}\right)^{1/2}. \tag{A.8}$$

We conclude that under condition (A.8) the paraboloid given by (A.7) produces a solution to (1.1).

Case 2. Assume $A \neq 0$. Then, according to Chebichev's theorem on the integration of binomial differentials (A.6) can be solved explicitly (in terms of elementary functions) if and only if $d = 2$. Taking $d = 2$ and solving (A.6) we get

$$v(r) = B - \frac{1}{2}r\sqrt{K_0 r^2 + A} - \frac{AK_0^{-1/2}}{2}\ln|r + K_0^{-1/2}\sqrt{r^2 K_0 + A}| =: B - \varphi_A(r),$$

where A, B are constants, $A \neq 0$ and

$$A \geq -R_0^2 K_0. \tag{A.9}$$

From the boundary condition on u, we get

$$B = h_0 + \varphi_A(R_0),$$

and hence

$$v(r) = h_0 + \varphi_A(R_0) - \varphi_A(r), \qquad r \geq R_0.$$

From the free boundary condition, we see that $A = \lambda_0^2 - K_0 r^2$, if $v(r) = 0$. Comparing this expression for A with (A.9) we get

$$\lambda_0^2 - K_0 r^2 \geq -K_0 R_0^2,$$

and finally

$$r \leq \left(R_0^2 + \frac{\lambda_0^2}{K_0}\right)^{1/2} =: R_{max}, \tag{A.10}$$

which shows that radius of the 0-level set of v must be bounded above by R_{max}. Thus, taking into account the monotonicity of v, we conclude that v gives a solution if and only if $v(R_{max}) \leq 0$. Plugging the value of A into φ we get

$$\varphi_A(x) = \frac{1}{2}r\lambda_0 + \frac{(\lambda_0^2 - K_0 r^2)}{2}K_0^{-1/2}\ln|r + K_0^{-1/2}\lambda_0| =: \varphi(r).$$

Hence, the function $u(x) = h_0 + \varphi(R_0) - \varphi(|x|)$ produces a solution to (1.1) if and only if

$$h_0 + \varphi(R_0) - \varphi(R_{max}) \leq 0. \tag{A.11}$$

Clearly the validity of this condition depends on parameters h_0, K_0, λ_0, and is not always satisfied.

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